Discrete-time volatility forecasting with persistent leverage effect and the link with continuous-time volatility modeling

Web appendix.

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1 Details on the estimation of quadratic variation

Estimation of total quadratic variation is accomplished with the two-scale estimator of Zhang et al. (2005). Denote by X_j the observations of the logarithmic price with $j = 1, \dots, n$, where n is the number of prices each day. For a given K (the length in ticks of the subsampling interval), we define TSRV as:

$$\mathsf{TSRV} = \left(1 - \frac{n-K+1}{nK}\right)^{-1} \left(\frac{1}{K}\sum_{j=1}^{n-K}(X_{j+K} - X_j)^2 - \frac{1}{K}\sum_{j=1}^{n-1}(X_{j+1} - X_j)^2\right)$$

where the first term is a small-sample correction. We use $K = 10$.

Implementation of TBPV and the C-Tz test follows closely Corsi et al. (2010), to which the interested reader is referred for a thorough discussion. The code is available from the authors upon request. Denote by $\Delta_{t,j}X = X_{j\delta+t} - X_{(j-1)\delta+t}$ the evenly sampled return at day t of the stochastic process at a frequency δ (we use $5-$minutes returns). We also write $\Delta_j X$ in place of $\Delta_{t,j}X$.

We first define iteratively a threshold function using a multiple of an estimator of the local variance:

$$\hat{V}_t^Z = \frac{\sum_{i=-L,\ i\neq -1,0,1}^{L} K\left(\frac{i}{L}\right) (\Delta_{t+i}X)^2\, I_{\{(\Delta_{t+i}X)^2 \le c_V^2 \cdot \hat{V}_{t+i}^{Z-1}\}}}{\sum_{i=-L,\ i\neq -1,0,1}^{L} K\left(\frac{i}{L}\right) I_{\{(\Delta_{t+i}X)^2 \le c_V^2 \cdot \hat{V}_{t+i}^{Z-1}\}}}, \qquad Z = 1, 2, \dots \tag{1.1}$$

with the starting value $\hat{V}^0 = +\infty$, which corresponds to using all observations in the first step, and $c_V = 3,\ L = 25$. We use a Gaussian kernel $K(y) = \left(1/\sqrt{2\,\pi}\right) \exp\left(-y^2/2\right)$. Iteration stops when $\hat{V}_t^Z = \hat{V}_t^{Z+1}$; we then set:

$$\theta_t = 9 \cdot \hat{V}_t^Z$$

This allows the computation of TBPV (we also use the notation $\vartheta_j = \vartheta_{j\delta}$).

To define the C-Tz test we need to introduce the corrected threshold multipower variation, defined as:

$$\text{C-TMPV}_\delta^{[\gamma_1,\ldots,\gamma_M]} = \delta^{1-\frac{1}{2}(\gamma_1+\ldots+\gamma_M)} \sum_{j=M}^{[T/\delta]} \prod_{k=1}^{M} Z_{\gamma_k}(\Delta_{j-k+1}X, \vartheta_{j-k+1}) \tag{1.2}$$

where the function $Z_\gamma(x, y)$ is defined as:

$$Z_\gamma(x,y) = \begin{cases} |x|^\gamma & if\ x^2 \le y \\ \frac{1}{2N(-c_\vartheta)\sqrt{\pi}} \left(\frac{2}{c_\vartheta^2} y\right)^{\frac{\gamma}{2}} \Gamma\left(\frac{\gamma+1}{2}, \frac{c_\vartheta^2}{2}\right) & if\ x^2 > y \end{cases} \tag{1.3}$$

The C-Tz test is then defined as:

$$\text{C-Tz} = \delta^{-\frac{1}{2}} \frac{(\text{RV} - \text{C-TBPV}_\delta) \cdot \text{RV}^{-1}}{\sqrt{\left(\frac{\pi^2}{4} + \pi - 5\right) \max\left\{1, \frac{\text{C-TTriPV}_\delta}{(\text{C-TBPV}_\delta)^2}\right\}}}, \tag{1.4}$$

where $\text{RV} = \sum_{j=1}^{[T/\delta]} (\Delta_j X)^2$, $\text{C-TBPV}_\delta = \mu_1^{-2}\, \text{C-TMPV}_\delta^{[1,1]}$ and $\text{C-TTriPV}_\delta = \mu_{\frac{4}{3}}^{-3} \cdot \text{C-TMPV}_\delta^{[\frac{4}{3},\frac{4}{3},\frac{4}{3}]}$, with $\mu_1 \simeq 0.7979$ and $\mu_{\frac{4}{3}} \simeq 0.8309$.

2 Simulation: The impact of jumps

With our simulation study we show that uncovering the presence of jumps in a reliable fashion is important for two distinct reasons. First, jumps have a direct impact on volatility dynamics which can be explained by the presence of contemporaneous component in price and volatility jumps. Put in other words, our finding suggests that jumps in price are typically associated with jumps in volatility. This result has also been found by Todorov and Tauchen (2011) with a different empirical investigation based on the VIX index sampled at high frequency; see also Jacod and Todorov (2010). Second, removing the jump component, which we find to be almost unpredictable, has a trimming effect on the volatility series, which allows a better fit of the model, a point already made by Andersen et al. (2007).

We simulate the stock index price with the parametric specification of Eraker et al. (2003), that is:

$$\begin{pmatrix} dX_t \\ dV_t \end{pmatrix} = \begin{pmatrix} \mu \\ \kappa(\theta - V_{t-}) \end{pmatrix} dt + \sqrt{V_{t-}} \begin{pmatrix} 1 & 0 \\ \sigma_v \rho & \sigma_v \sqrt{1-\rho^2} \sigma_v \end{pmatrix} dW_t + \begin{pmatrix} \xi^y dN_t^y \\ \xi^v dN_t^V \end{pmatrix} \tag{2.1}$$

where W_t is a bivariate Brownian motion and dN^y and dN^V are Poisson processes with intensity λ^y and λ^V respectively; ξ^y is normally distributed, while ξ^V has an exponential law. As in Eraker et al. (2003), we consider two cases: the case in which dN^y is independent from dN^V (what they name the SVIJ model) and the case in which $dN^y = dN^V$ (what they name the SVCJ model), and we hold their terminology. We use exactly the parameters estimated by Eraker et al. (2003) for the S&P500 time series and simulate $6,000$ days. Figure 1 and 2 report the results.

In the SVIJ case, jumps have no impact on future volatility, but there is still a benefit in removing the jump component. Indeed, in this model the persistence is conveyed only by the continuous volatility, while total quadratic variation (which is estimated by realized volatility) also contain the memoryless jumps. Thus, by separating the jumps from the persistent part in the explanatory variables, a better model specification is obtained. We conclude that, when the memory of volatility is mainly contained in the continuous part of quadratic variation, there is still a potential benefit in removing jumps even if they do not impact on future volatility. This benefit persist also for long horizon forecasts. Importantly, in this case, the jump component is found to be insignificant.

In the SVCJ model, when a jump occurs in price it also occurs in volatility and it is positive. Thus, when there is a jump in price, volatility becomes higher and it stays higher because of its memory persistence. That explains why, in this case, jumps are found to be significant in explaining future volatility, contrary to the SVIJ case. Hence, our simulation results show that

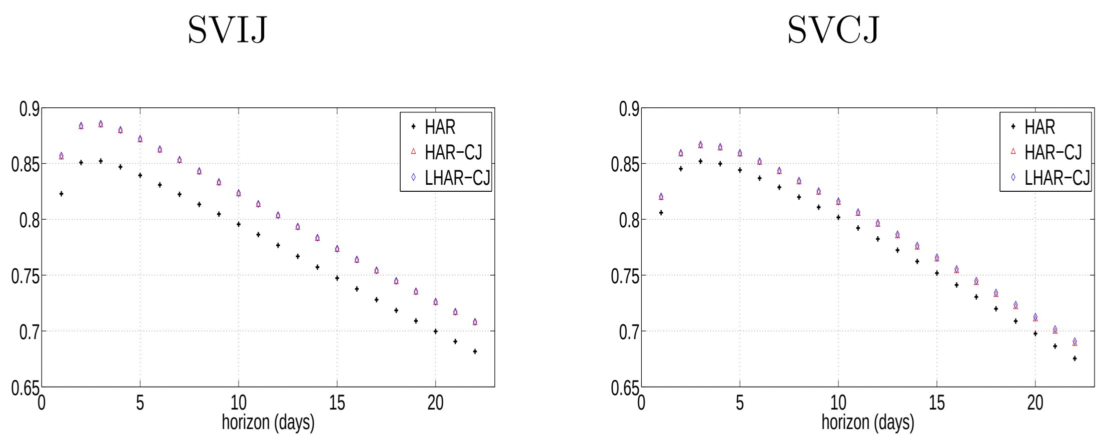

Figure 1: R^2 of Mincer-Zarnowitz regressions for realized volatility forecast ranging from 1 day to 1 month of 6,000 days simulated data from SVIJ (left) and SVCJ(right) model. The forecasting models are the standard HAR with only heterogeneous volatility, the HAR-CJ with heterogeneous jumps and the LHAR-CJ model.

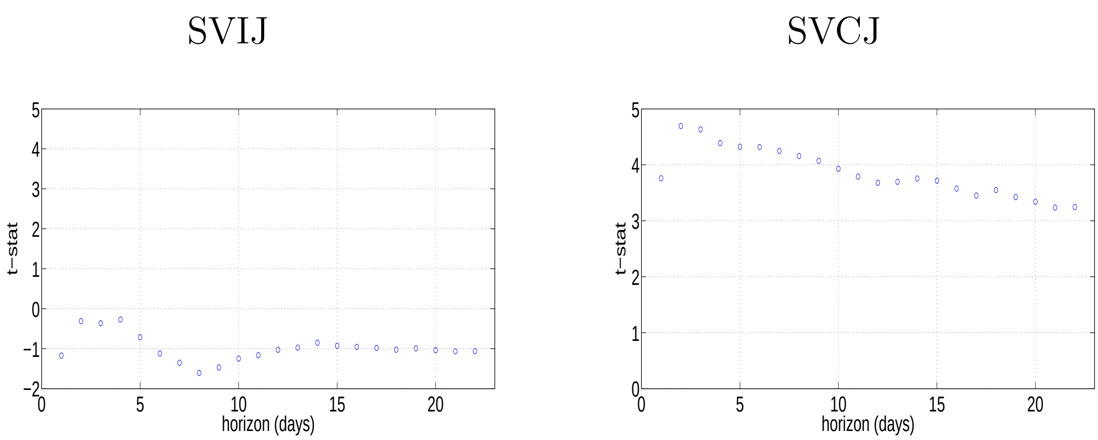

Figure 2: t-statistics of daily jump coefficients for LHAR-CJ model estimated on 6,000 simulated daily data from SVIJ (left) and SVCJ (right) model as a function of the forecasting horizon h.

	SVIJ regression					SVCJ regression			
	1 day	1 week	2 weeks	1 month		1 day	1 week	2 weeks	1 month
c	0.235*	0.441*	0.678*	1.220*	c	0.291*	0.521*	0.814*	1.476*
	(6.325)	(6.816)	(6.885)	(7.301)		(6.364)	(7.301)	(7.817)	(8.131)
$\widehat{\mathsf{C}}$	0.638*	0.594*	0.549*	0.479*	$\widehat{\mathsf{C}}$	0.440*	0.429*	0.398*	0.370*
	(16.288)	(15.437)	(14.403)	(14.852)		(12.103)	(11.229)	(10.495)	(10.797)
$\widehat{\mathsf{C}}^{(5)}$	0.341*	0.340*	0.345*	0.339*	$\widehat{\mathsf{C}}^{(5)}$	0.537*	0.540*	0.555*	0.516*
	(7.861)	(7.691)	(6.817)	(6.022)		(13.527)	(10.555)	(9.306)	(7.375)
$\widehat{\mathsf{C}}^{(22)}$	-0.024	-0.017	-0.022	-0.047	$\widehat{\mathsf{C}}^{(22)}$	-0.036*	-0.074*	-0.117*	-0.182*
	(-1.516)	(-0.746)	(-0.595)	(-0.836)		(-2.048)	(-2.453)	(-2.811)	(-2.880)
$\widehat{\mathsf{J}}$	-0.012	-0.006	-0.011	-0.009	$\widehat{\mathsf{J}}$	0.042*	0.044*	0.040*	0.038*
	(-1.174)	(-0.713)	(-1.249)	(-1.062)		(3.848)	(4.288)	(3.924)	(3.154)
$\widehat{\mathsf{J}}^{(5)}$	-0.001	-0.001	0.014*	0.012	$\widehat{\mathsf{J}}^{(5)}$	0.019*	0.019*	0.023*	0.022*
	(-0.296)	(-0.120)	(2.184)	(1.376)		(4.922)	(4.306)	(3.928)	(3.487)
$\widehat{\mathsf{J}}^{(22)}$	-0.000	-0.001	-0.004	-0.009	$\widehat{\mathsf{J}}^{(22)}$	0.006*	0.007	0.008	0.010
	(-0.206)	(-0.235)	(-0.694)	(-1.118)		(2.080)	(1.720)	(1.363)	(1.071)
r^-	-0.052*	-0.038*	-0.039*	-0.041*	r^-	-0.042*	-0.040*	-0.038*	-0.038*
	(-5.444)	(-5.417)	(-5.496)	(-5.224)		(-4.277)	(-5.308)	(-4.829)	(-4.347)
$r^{(5)-}$	-0.016	-0.031	0.001	0.018	$r^{(5)-}$	-0.029	-0.024	-0.038	-0.065
	(-0.994)	(-1.474)	(0.048)	(0.598)		(-1.388)	(-0.893)	(-1.390)	(-1.754)
$r^{(22)-}$	0.040	0.057	0.033	0.002	$r^{(22)-}$	0.079	0.053	0.025	-0.002
	(1.651)	(1.603)	(0.653)	(0.029)		(1.569)	(0.835)	(0.320)	(-0.015)
R^2	0.8567	0.8718	0.8232	0.7076	R^2	0.8197	0.8587	0.8152	0.6895
HRMSE	0.1325	0.1448	0.1661	0.2033	HRMSE	0.1254	0.1298	0.1460	0.1790

Table 1: OLS estimate for baseline LHAR-CJ model, for a time series of 6,000 days simulated with models SVIJ and SVCJ of Eraker et al. (2003).

a possible mechanism explaining the significant impact of jumps on future volatility is given by the presence of contemporaneous jumps in price and volatility, a possibility which has been recently empirically confirmed by Todorov and Tauchen (2011). It is remarkable the similarity between the figures reporting the Newey-West corrected t-statistics of the daily jump coefficient estimated on the simulated SVCJ model (Figure 2 right panel) and on the empirical S&P500 (Figure 3).

On the other hand, the heterogeneous leverage effect found in real data cannot be completely explained by model (2.1). Indeed, the presence of a negative coefficient $\rho \approx -0.5$ (estimated on S&P 500 data) is able to explain only short-period leverage effect, by propagating negative returns into contemporaneous, and by memory persistence, future volatility; while, in the real data, we provided evidence for strong persistence in the leverage effect, being also the weekly and monthly negative components highly significant. The model specification 2.1 is then insufficient to explain our results which demand for a more complicated continuous process with a richer specification.

3 Additional empirical results

Table 2 reports the results of the LHAR-CJ regressions in which the leverage terms are constructed with daily open to close returns, i.e. without the contribution of the overnight returns.

Figure 3 plots the t-statistics of the impact of the daily jump on aggregated volatility at different time horizons, confirming, with its rapid decline, that daily jumps affects future volatilities much strongly over a short period of about one week, even if it remains highly significant at all the considered horizons.

For the residuals of the different HAR type of models we confirm the results of Corsi et al. (2008) finding no significant autocorrelation but some remaining heteroskedasticity.

In the Diebold-Mariano tests we employ the Clark and West adjustment whenever the two models are nested i.e., in our case, when comparing HAR-CJ with LHAR-CJ. In the HAR vs. (L)HAR-CJ comparison, however, the models are not nested since the (L)HAR-CJ models contain the jump components while the HAR does not.

4 Implementation of indirect inference

Indirect inference is a simulation-based method for estimating the parameters of a *structural model* (Gourieroux and Monfort, 1996). The structural model we use in this paper are

S&P500 LHAR in-sample regression, period 1982–2009

Variable	One day	One week	Two weeks	One month
c	0.395*	0.497*	0.613*	0.826*
	(9.914)	(8.412)	(7.914)	(7.437)
$\widehat{\mathsf{C}}$	0.310*	0.210*	0.164*	0.125*
	(17.332)	(14.559)	(13.419)	(10.543)
$\widehat{\mathsf{C}}^{(5)}$	0.376*	0.364*	0.335*	0.284*
	(14.469)	(11.576)	(9.232)	(6.484)
$\widehat{\mathsf{C}}^{(22)}$	0.220*	0.316*	0.367*	0.414*
	(10.746)	(10.830)	(9.993)	(9.135)
$\widehat{\mathsf{J}}$	0.042*	0.020*	0.018*	0.013*
	(6.985)	(4.611)	(4.719)	(4.135)
$\widehat{\mathsf{J}}^{(5)}$	0.010*	0.013*	0.011*	0.010
	(3.322)	(3.060)	(2.214)	(1.893)
$\widehat{\mathsf{J}}^{(22)}$	0.005*	0.008*	0.010*	0.014*
	(2.201)	(2.080)	(2.185)	(2.335)
r^{-}	-0.009*	-0.005*	-0.004*	-0.003*
	(-12.182)	(-10.848)	(-8.949)	(-5.296)
$r^{(5)-}$	-0.007*	-0.005*	-0.007*	-0.007*
	(-4.166)	(-2.511)	(-3.430)	(-3.280)
$r^{(22)-}$	-0.007*	-0.010	-0.008	-0.004
	(-1.973)	(-1.734)	(-1.085)	(-0.400)
R^2	0.7681	0.8129	0.8018	0.7620
HRMSE	13.6323	10.6361	10.8124	12.4990

Table 2: OLS estimates of LHAR-CJ regressions with leverage terms constructed with daily open to close returns for the S&P500 futures from 28 April 1982 to 5 February 2009 ($6,669$ observations). The LHAR-CJ model is estimated with $h = 1$ (one day), $h = 5$ (one week), $h = 10$ (two weeks) and $h = 22$ (one month). The significant jumps are computed using a critical value of $\alpha = 99.9\%$. Reported in parenthesis are t-statistics based on Newey-West correction with $L = 2 + 2h$ number of lags and Bartlett kernel. A star denotes 95% significance.

continuous-time models. An estimate of the parameters is obtained using a parametric *auxiliary model* (in this paper: the HAR and LHAR model). Denote by θ the parameter vector of the auxiliary model and by ψ the parameter vector of the structural model. We first estimate the auxiliary model on the data, and denote by $\widehat{\theta}$ the estimated parameter vector. Then, for a given ψ, we produce $\overline{M} = 100$ simulated replicas (with antithetic variates) of the structural models. Exact simulation of the fractional Brownian motion is accomplished using the circulant embedding method of Yingchun Jasmine Zhou and Stilian Stoev and available on their web page. In

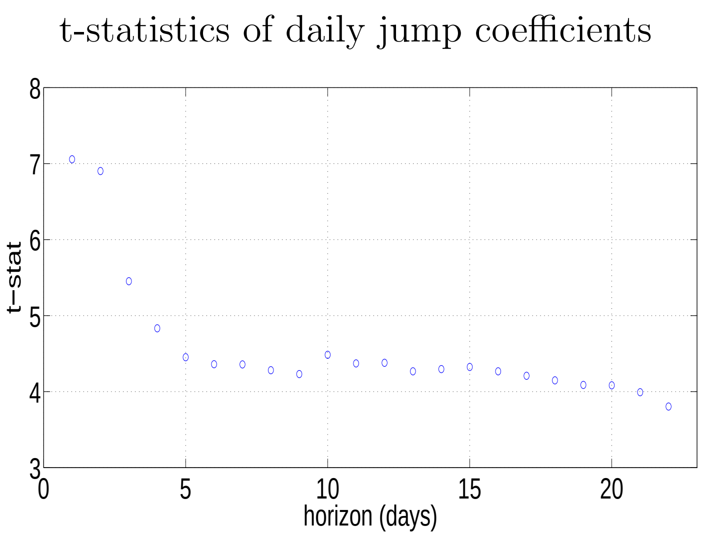

Figure 3: t-statistics of daily jump coefficients for LHAR-CJ model estimated on S&P500 from 28 April 1982 to 5 February 2009 ($6,669$ observations excluding the October 1987 crash) as a function of the forecasting horizon h .

our case, we simulated time series of $\overline{n} = 7,000$ days, and on each day we simulate $M^* = 80$ intraday returns. Since in our simulated setting there is no microstructure noise, we estimate quadratic variation using realized variance, that is sum of intraday squared returns. With these time series of $\overline{n} = 7,000$ realized variances, we estimate the auxiliary model (HAR or LHAR) on each replica; estimates are denoted by $\theta_j^*(\psi)$, with $j = 1, \ldots, \overline{M}$. The structural parameter vector is then estimated by $\widehat{\psi} = \arg\min \chi^2$ where

$$\chi^2 = \left(\sum_{j=1}^{\overline{M}} \left(\widehat{\theta} - \theta_j^*(\psi) \right) \right)' W \left(\sum_{j=1}^{\overline{M}} \left(\widehat{\theta} - \theta_j^*(\psi) \right) \right)$$

where W is a suitable positive-definite weighting matrix, which we set equal to the variance-covariance matrix of the parameters of the auxiliary model estimated on the data. Implied parameters of the auxiliary model are then averages of $\theta_j^*(\widehat{\psi})$.

When using as auxiliary model the LHAR-CJ model, the computational time required becomes huge. To save time, we estimate the (unfeasible) case in which we know when a jump occurs and what is is size, and the disentangling of continous and discontinuous quadratic variation proceeds accordingly.

When jumps are not considered, even if we refer to the HAR and LHAR model, we actually use the time series of $\widehat{\mathsf{C}}_t$ in our samples as our observables, and the estimates of the (L)HAR-C model.

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