ARBITRARY INITIAL TERM STRUCTURE WITHIN THE CIR MODEL: A PERTURBATIVE SOLUTION

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ABSTRACT. Single-factor interest rate models with constant coefficients are not consistent with arbitrary initial term structures. An extension which allows both arbitrary initial term structure and analytical tractability has been provided only in the Gaussian case.
In this paper, within the context of the HJM methodology, we provide an extension of the CIR model which admits arbitrary initial term structure. We show how to calculate bond prices via a perturbative approach, and we provide closed formulas at every order. Since the parameter we select for the expansion is typically estimated to be small, the perturbative approach turns out to be adequate to our purpose. Using results on affine models, we estimate the extended CIR model via maximum likelihood on a time series of daily interest rate yields. Our results show that the CIR model has to be rejected with respect to the proposed extension, and point out that the extended CIR model provides a more flexible characterization of the link between risk neutral and natural probability.

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1. INTRODUCTION

It is well known that, in spite of their popularity, single-factor interest rate models with constant coefficients are not consistent with arbitrary initial term structures. To tackle this problem, Hull and White (1990, 1993) showed that an extension of the Vasicek (1977) model allowing for arbitrary initial term structure can be provided and dealt with analytically. Following these results, they outlined a possible way to overcome this difficulty for a larger class of models, including the square-root diffusion of the CIR model (Cox et al., 1985a,b): namely they suggested to modify the drift of the spot rate dynamics in order to include time depending terms accounting for the initial term structure. Due to the mathematical complexity of the approach, the Authors proved only the feasibility of the extension, but they did not give any information about neither the analytical features of the dynamics of the spot rate, nor the bond pricing problem.

In this paper we discuss, in the context of the Heath-Jarrow-Morton (HJM) methodology (Heath et al., 1990, 1992), an extension of the CIR model which is consistent with arbitrary initial term structures.

In the first part we provide a complete characterization of the model by specifying the dynamics of the bond prices and of the spot rate. We discuss the solution of the bond pricing problem and we show that the analytical form of the pricing formula depends on the solution of a Volterra integral equation of the first kind.

The dynamics of the spot rate is also derived; to fit any observable term structure, the drift term cannot be arbitrary, but it must depend on the initial term structure. It will be shown how to include the initial term structure into the drift term, thus obtaining the analytical form of the spot rate dynamics. It is characterized by a mean reverting diffusion process with time-dependent reversion level and constant mean reversion rate, and it still belongs to the affine class of Duffie and Kan (1996).

The solution of the Volterra equation affects both the dynamics of the spot rate and the bond pricing formula. Exact solutions are not straightforward and we solve the Volterra equation following a perturbative approach in which the volatility coefficient of the spot rate dynamics is used as perturbation parameter, thus providing closed formulas at every order. Since the parameter we select for the expansion is typically estimated to be small for interest rate data, the perturbative approach, and in particular the first order solution, is supposed to be adequate for practical purposes.

In the second part of the paper, borrowing from affine process theory, we estimate the extended CIR model via maximum likelihood on a time series of daily zero-coupon yields, namely the three-month Euribor. Our results show that the CIR model with constant coefficients has to be rejected with respect to the extension, pointing out that the extended CIR model provides a more reliable characterization of time series data, as well as a more flexible characterization of the link between risk neutral and natural probability.

The remainder of the paper is organized as follows. After a brief description in Section 2 of the theoretical framework and of the main results which will be used in our work, in Section 3 we illustrate in some details the perturbation method used to solve the Volterra equation and to derive the spot rate dynamics. The model is estimated in Section 4. Finally, some comments conclude the paper.

2. SOME BASIC RESULTS

In this Section, we start by recalling some basic results. We work in a HJM framework and we denote by,

$$P(r(t),t;T) = \exp\left(-\int_t^T f(r(t),t,u)du\right), \tag{2.1}$$

the price at time t of a T-maturing pure discount bond. We denote by $f(r(t),t,T)$ the forward rate curve, and $r(t) = f(r(t),t,t)$ the spot rate.
We assume that $P(r(t),t;T)$, or equivalently $f(r(t),t,T)$, are smooth functions of their arguments[1].
The model we propose is described by the following Cauchy problem,

$$\begin{aligned} \frac{dP}{P}(r(t),t;T) &= r(t)dt - \sqrt{kr(t)}B(t,T)dw(t) \\ P(r(0),0;T) &= P^*(0,T), \end{aligned} \tag{2.2}$$

[1]The term smooth is used here to mean that the discount function $P(r,t;T)$ or equivalently the forward rate $f(r,t;T)$, are continuous and twice differentiable with respect to all their arguments.

which defines the dynamics of zero-coupon bonds prices under the risk-neutral measure. $P^*(0,T)$ is the initial term structure[2], k is a parameter, and $w(t)$ a Wiener process. $B(t,T)$ is given by

$$B(t,T) = \frac{e^{d(T-t)} - 1}{\phi\left[e^{d(T-t)} - 1\right] + d}, \tag{2.3}$$

where

$$\phi = \frac{1}{2}\left[d + \sqrt{d^2 - 2k}\right]. \tag{2.4}$$

It is easy to verify that the volatility structure of zero-coupon bond returns,

$$\sigma_p(r(t),t;T) = \sqrt{kr(t)}B(t,T), \tag{2.5}$$

coincides with the CIR volatility structure.

It has been shown (Jeffrey, 1995; Mari, 2002) that the above model is consistent with any initial term structure $P^*(0,T)$, and that the spot rate follows a well defined Markov process. Furthermore, a closed form characterization of the term structure can be given. In fact, if the volatility structure of zero-coupon bond returns satisfies the CIR condition, the solution of the Cauchy problem is,

$$\begin{aligned} P(r(t),t;T) =& \frac{P^*(0,T)}{P^*(0,t)} \exp\left[f^*(0,t)B(t,T) + \right. \\ & \left. - \int_0^t H(u)B(u,T)du - \frac{k}{2}\int_0^t f^*(0,u)B^2(u,T)du\right] e^{-r(t)B(t,T)}, \end{aligned} \tag{2.6}$$

where $H(t)$ satisfies the following Volterra integral equation of the first kind,

$$\int_0^t H(u)B(u,t)du = G(t), \tag{2.7}$$

with

$$G(t) = -\frac{k}{2}\int_0^t f^*(0,u)B^2(u,t)du, \tag{2.8}$$

and $f^*(0,T)$ is the initial forward rate curve (Mari, 2002).

As it is easy to verify by posing $t = 0$, the functional form (2.6) of the bond pricing formula is consistent with any observable term structure. It also depends on the solution of an associated Volterra integral equation of the first kind. In general exact solutions are not

[2] $P^*(0,T)$ is also assumed to be continuous and twice differentiable with respect to maturity.

straightforward and numerical methods are to be used. We propose in the following Section a perturbative solution of the Volterra equation which can be very useful for applications.

As we pointed out above, the dynamics of the spot rate can be derived from the bond pricing formula (2.6) and is described by the Markov process given below,

$$dr(t) = \big[a(t) - (2\phi - d)r(t)\big]dt + \sqrt{kr(t)}dw(t), \tag{2.9}$$

where

$$a(t) = \frac{\partial f^*(0,t)}{\partial t} + (2\phi - d)f^*(0,t) - H(t). \tag{2.10}$$

The dynamics of $r(t)$ is characterized by a mean reverting Markov process in which both the drift and the diffusion coefficients are affine in the spot rate. Since the model is consistent with arbitrary initial term structures, the drift coefficient cannot be time independent but it must related to the market data via equation (2.10). The drift is still characterized by a time-dependent reversion level. The mean reversion rate $2\phi - d$, is constant and coincides with the CIR mean reversion rate parameter. The solution of the Volterra equation affects both the bond pricing formula and the spot rate dynamics. In the following Section we solve the Volterra equation following a perturbative approach in which k, the volatility coefficient of the spot rate dynamics, is used as perturbation parameter.

3. THE SOLUTION OF THE VOLTERRA EQUATION

Under the CIR volatility assumption, we will prove in this Section that the Volterra equation can be solved by using perturbation methods. Let us assume that $H(t)$ can be expanded in a power series in the volatility parameter k around the value $k = 0$ as follows,

$$H(t) = H_0(t) + H_1(t)k + H_2(t)k^2 + \cdots = \sum_{j=0}^{+\infty} H_j(t)k^j. \tag{3.1}$$

The expansion of $B(t,T)$ can be easily obtained getting,

$$B(t,T) = B_0(t,T) + \frac{B_0^2(t,T)}{2d}k + \frac{B_0^2(t,T) + dB_0^3(t,T)}{4d^3}k^2 + \cdots\,, \tag{3.2}$$

where

$$B_0(t,T) = \frac{1}{d}\left[1 - e^{-d(T-t)}\right], \tag{3.3}$$

The higher order coefficients are also polynomials in B_0. Substituting (3.1), (3.2) into (2.7), and equating the coefficients order by order in the expansion, we get,

$$\int_0^t H_j(u)B_0(u,t)du = G_j(t), \qquad j = 0,1,2,\cdots. \tag{3.4}$$

It can be easily verified that since $G_0(t) = 0$, then $H_0(t) = 0$, so that the first two terms are given by,

$$G_1(t) = -\frac{1}{2}\int_0^t f^*(0,u)B_0^2(u,t)du, \tag{3.5}$$

$$G_2(t) = -\frac{1}{2d}\int_0^t \Big[f^*(0,u)B_0(u,t) + H_1(u)\Big]B_0^2(u,t)du. \tag{3.6}$$

The higher order terms can be determined in a similar way.

All equations (3.4) are characterized by the same analytical structure. At any order in the perturbative expansion, the right hand side, $G_j(t)$ is a known function, depending on B_0, $f^*(0,t)$ and on the solutions of the previous equations $H_i(t)$ $i = 0,1,2,\cdots,j-1$. Although (3.4) are Volterra integral equations of the first kind, in this case the solutions can be calculated exactly by differentiating twice with respect to t both members of any equation. Since $B_0(t,t) = 0$, after the first differentiation we get,

$$\int_0^t e^{du}H_j(u)du = e^{dt}G_j'(t), \qquad j = 0,1,2,\cdots, \tag{3.7}$$

and after the second we obtain the final result,

$$H_j(t) = dG_j'(t) + G_j''(t), \qquad j = 0,1,2,\cdots. \tag{3.8}$$

Up to the second order in k the solutions are given by,

$$H_0(t) = 0, \tag{3.9}$$

$$H_1(t) = -e^{-2dt}\int_0^t e^{2du}f^*(0,u)du, \tag{3.10}$$

and

$$H_2(t) = -\frac{e^{-2dt}}{d}\int_0^t e^{2du}\Big[3f^*(0,u)B_0(u,t) + H_1(u)\Big]du. \tag{3.11}$$

The higher order solutions can be found in a similar way.

We then provided closed form solution, iteratively calculated, for any order of the perturbative expansion. Typical estimates of the volatility parameter k on interest rate data are small, e.g. Chan et al. (1992) find an annualized value of $k = 0.073$. Thus, we are quite confident that high order solutions affect pricing negligibly. In the following Section, we estimate the model at the first order of the perturbation scheme.

4. ESTIMATING THE MODEL

In this Section we propose an estimate of the extended CIR model up to the first order in the perturbative expansion, and we determine the parameters of the spot rate process. We assume that the market price of risk is of the following form (Cox et al., 1985a,b),

$$q(r(t)) = \frac{\pi}{\sqrt{k}}\sqrt{r(t)}, \tag{4.1}$$

so that the standard CIR model under the natural probability reads,

$$dr(t) = [a - (2\phi - d + \pi)r(t)]\,dt + \sqrt{kr(t)}dw^*(t), \tag{4.2}$$

where

$$\phi = \frac{1}{2}\big[d + \sqrt{d^2 - 2k}\big]. \tag{4.3}$$

Four parameters are to be determined: d, k, π and a. On the other side, the extended version of the CIR model (up to the first order in the perturbative expansion) is given by,

$$dr(t) = \big[a(t) - (2\phi - d + \pi)r(t)\big]dt + \sqrt{kr(t)}dw^*(t), \tag{4.4}$$

where

$$a(t) = \frac{\partial f^*(0,t)}{\partial t} + (2\phi - d)f^*(0,t) - H(t), \tag{4.5}$$

and

$$H(t) = -ke^{-2dt}\int_0^t e^{2du} f^*(0,u)du. \tag{4.6}$$

In this case three parameters (d, k, and π) must be determined in addition to the initial forward rate curve $f^*(0,t)$.

It is important to point out in what sense we mean that the initial term structure can be arbitrary. Indeed, Jeffrey (1995) shows that, denoting by $\gamma(r,t,T)$ the derivative of the bond volatility structure σ_p with respect to the maturity T,

$$\gamma(r,t,T) = \frac{\partial \sigma_p(r,t,T)}{\partial T}, \tag{4.7}$$

the initial term structure can be cast in the following form,

$$f^*(r,0,T) = \int_0^r \frac{\gamma(s,0,T)}{\gamma(s,0,0)} ds + \eta(T), \tag{4.8}$$

where $\eta(0) = 0$. The arbitrariness of the initial term structure is all contained in the function $\eta(T)$. In our case, since $\sigma_p(r,t,T) = \sqrt{kr(t)}B(t,T)$, we have,

$$f^*(r,0,T) = \left[\frac{de^{dT/2}}{\phi(e^{dT}-1)+d}\right]^2 r + \eta(T). \tag{4.9}$$

Actually, not all the choices of $\eta(T)$ provide economic significance to the initial term structure; for example, it is reasonable to conjecture that, $\lim_{T\to+\infty} f^*(0,T)$ exists and it is finite, and that

$$\int_0^{+\infty} f^*(0,s)ds = +\infty, \tag{4.10}$$

which ensures that one dollar to be received at infinite time to maturity is worth nothing today. Following the above considerations, we introduce a parametric form for the initial term structure in the spirit of Nelson and Siegel (1987) specifying it as,

$$f^*(0,T) = r(0)\left[\frac{de^{dT/2}}{\phi(e^{dT}-1)+d}\right]^2 + \eta_1\left(1-e^{-\xi T}\right) + \eta_2 T e^{-\xi T}. \tag{4.11}$$

We clearly have $f^*(0,0) = r(0)$, and $\lim_{T\to+\infty} f^*(0,T) = \eta_1$. In this sense we can interpret η_1 as the infinite maturity yield. The parameter ξ calibrates the velocity at which the yield curve tends to its limit η_1, while η_2 allows for convexity change in the initial term structure. The need for at least three factors can be motivated by the earlier findings of Litterman and

Scheinkman (1991), who analyzed by principal components a huge number of yield curves, and concluded that three factors are good enough to account for all the empirical properties of the observed yield curve. From this point of view we can interpret η_1, ξ, η_2 as parameters tuning, respectively, the level, the slope and the curvature of the yield curve, see also the discussion in Andersen and Lund (1997).

In our estimate, we will determine therefore three additional parameters, η_1, ξ, η_2, thus finding the initial term structure which is compatible with the observed time series. Clearly, in the operational practice, a different point of view may be adopted: one can fit the three parameters cross-sectionally, and the spot rate parameters on the time series, in order to infer forecasts or derivative prices.

We will estimate the extended CIR model efficiently via maximum likelihood, using the first-order perturbative approximation. Estimation via maximum likelihood can be accomplished since our model is affine, thus we can compute the transition density via inversion of the characteristic function, as suggested in Singleton (2001).
The characteristic function of a univariate process X_t is defined by,

$$\varphi_{X_t}(u;t,T) = \mathbb{E}_t^{\mathcal{Q}}\left[e^{iu\cdot X_T}\right]. \tag{4.12}$$

It is easy to check that it is the Fourier transform of the transition density, so that we can obtain the latter by inversion,

$$f(X_{t+1}|X_t) = \frac{1}{\pi}\int_0^{+\infty} Re\left[e^{-iu\cdot X_{t+1}}\varphi_{X_t}(u)\right] du. \tag{4.13}$$

In our case, the diffusion is univariate and we have not the problem of the curse of dimensionality which, for example, is found in Mari and Renò (2001), who adopt the same approach to estimate the parameters of an affine model for credit risk.

Since the model belongs to the affine class, the following exponential representation of the characteristic function holds,

$$\varphi_{r_t}(u;t,T) = e^{\alpha(u;t,T)+\beta(u;t,T)r_t}, \tag{4.14}$$

(see e.g. Duffie et al. (2002)), where α and β solve the following system of ordinary differential equations,

$$\begin{aligned} \frac{\partial \alpha(u;t,T)}{\partial t} &= -a(t)\beta(u;t,T) \\ \frac{\partial \beta(u;t,T)}{\partial t} &= (2\phi - d + \pi)\beta(u;t,T) - \frac{1}{2}k\beta^2(u;t,T) \end{aligned} \tag{4.15}$$

with the boundary condition $\alpha(u;T,T) = 0$, $\beta(u;T,T) = iu$. The solution for β is easily obtained,

$$\beta(u;t,T) = \frac{2iu(2\phi - d + \pi)}{e^{(2\phi-d+\pi)(T-t)}\left[2(2\phi - d + \pi) - iuk\right] + iuk}, \tag{4.16}$$

while α can be computed via integration,

$$\alpha(u;t,T) = \int_t^T a(s)\beta(u;s,T)ds. \tag{4.17}$$

Thus, the characteristic function is easily computable via a straightforward numerical integration (4.17).

One problem is the fact that we observe the bond price instead of $r(t)$; anyway the two are linked by the exponential-affine relation (2.6). As in Mari and Renò (2001), we will use $p(r(t),t;T)$ as working variable, defined as follows

$$p(r(t),t;T) = \log P(r(t),t;T) = A_0(t,T) + A_1(t,T)r(t), \tag{4.18}$$

where A_0 and A_1 are inferred from (2.6). Since the change of variables is affine, we preserve the exponential-affine structure of the characteristic function. We have, indeed,

$$\varphi_{p_t}(u;t,T) = e^{iuA_0(t,T)}\varphi_{r_t}(A_1(t,T)u;t,T). \tag{4.19}$$

We finally compute the transition density by,

$$f[p_{t_{i+1}}|p_{t_i}] = \frac{1}{\pi}\int_0^{+\infty} du\ Re\left[e^{-iup_{t_{i+1}}}e^{iuA_0}\varphi_{r_{t_i}}(A_1u;t_i,t_{i+1})\right]. \tag{4.20}$$

It is worth to note that also the integration to be performed in (4.20) is numerically straightforward, as the integrations to be performed to compute A_0, A_1.

Summarizing, we can compute the transition density of our process as a function of the six parameters $d, k, \pi, \eta_1, \xi, \eta_2$, in a quasi-analytical way, where the quasi stays for numerical integration. We perform all the numerical integrations via Gauss-Legendre quadrature, and

TABLE 1. Parameter estimates of the CIR model. Standard errors are estimated by $diag(\sqrt{-H^{-1}})$, where H is the Hessian as computed by numerical derivatives.

Log-Likelihood = 2044.264		
Parameter	Estimate	Standard Error
k	0.01320	(0.00013)
a	0.2651	(0.0021)
z	5.6347	(0.0065)
π	-1.85	(0.52)

we have a natural control on our results, i.e. the number of points used in the integration procedure. Instead of d, we use the parameter z defined as,

$$z = \sqrt{d^2 - 2k}. \tag{4.21}$$

z coincides with the mean reversion rate parameter in the risk-neutral probability. In this way, we do not have to impose the non-linear constraint $d^2 \geq 2k$, but the much simpler $z \geq 0$. We also impose $k \geq 0, \xi \geq 0, \eta_1 \geq 0$, together with the non-negativity of $f^*(0,T)$ for every T.

Our data set consists of the daily annualized yield of the three-month Euribor, from December, 30^{th}, 1998 to October, 18^{th}, 2001 for a total of 732 observations. We start by fitting the standard CIR model, which provides a natural benchmark to the extended CIR model. Parameter estimates are provided in Table 1. They imply a long-run mean of nearly 7% which is consistent with our data. The market price of risk is negative and significant.

We then fit the extended CIR model. We use two different ways to choose the number of points for integration. For the integral (4.20), which is the main focus of maximum likelihood estimation, we iterate the integration increasing the number of points until the change in the value of the integral is below a given small value. For all other integrals, we use a fixed number of points, and we increase them until we get stability on the results. We find that approximately 15 points are enough for the integral (4.20), while we get sufficient stability with 30 points for all the other integrals. Results are given in Table 2. The first thing to notice is that, on the basis of a likelihood-ratio test, we overwhelmingly reject the

TABLE 2. Parameter estimates of the extended CIR model. Standard errors are estimated by $diag(\sqrt{-H^{-1}})$, where H is the Hessian as computed by numerical derivatives.

Log-Likelihood = 2126.11		
Parameter	Estimate	Standard Error
k	0.00328	(0.00012)
z	5.071	(0.050)
π	7.74	(0.32)
η_1	0.1093	(0.0091)
ξ	0.4929	(0.0093)
η_2	0.0386	(0.0079)

CIR model with respect to the extended CIR model: the LR test value is 163.69 with 2 degrees of freedom. The failure of one-factor model, in particular the CIR model, to account for the observed interest rate time series has been widely documented in the literature, see Gentile and Renò (2002) for an example on Italian data. Thus, the rationale for extending the CIR model is now encouraged by the empirical results.

The estimate of z is consistent in the two estimates, while k is smaller (nearly one fourth) in the extended CIR case. In both case, the estimate of k is very small, thus the perturbative approach seems to be adequate. It is important to note that the market price of risk change sign in the new fit. Indeed, the market price of risk is the only parameter in the CIR model formulation which carries information on the link between the risk neutral and the natural probability. Given the rigidity of the yield curve in the CIR model, the estimate of the market price of risk could be unreliable. In the extended CIR framework, the specification of the yield curve is much more flexible, and in principle we admit arbitrary term structure. Thus, we find a more reasonable realization of the market price of risk.

5. CONCLUSIONS

In this paper, we propose an extension of the CIR model which preserves the CIR volatility structure. We look for bond pricing via perturbation methods, and we provide closed formula at any order in the expansion. The dynamics of the spot rate is also derived in a closed form. We then fit the extended CIR model on time series of interest rate data, and, based on likelihood ratio statistics, we show that it provides a much better description of the data.

The possibility to obtain the functional form of the dynamics of the spot rate greatly simplifies the problem of valuing interest rate contingent claims in the sense that we can use a partial differential equation representation of asset prices. Furthermore, since the model is consistent with arbitrary initial term structures, the valuation of contingent claims can be made consistent with observed bond prices.

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