Threshold estimation of jump-diffusion models and interest rate modeling

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Joint work with Cecilia Mancini, Università di Firenze.

Overlay of the presentation

- The econometric problem

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- Review of the literature

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- The proposed solution

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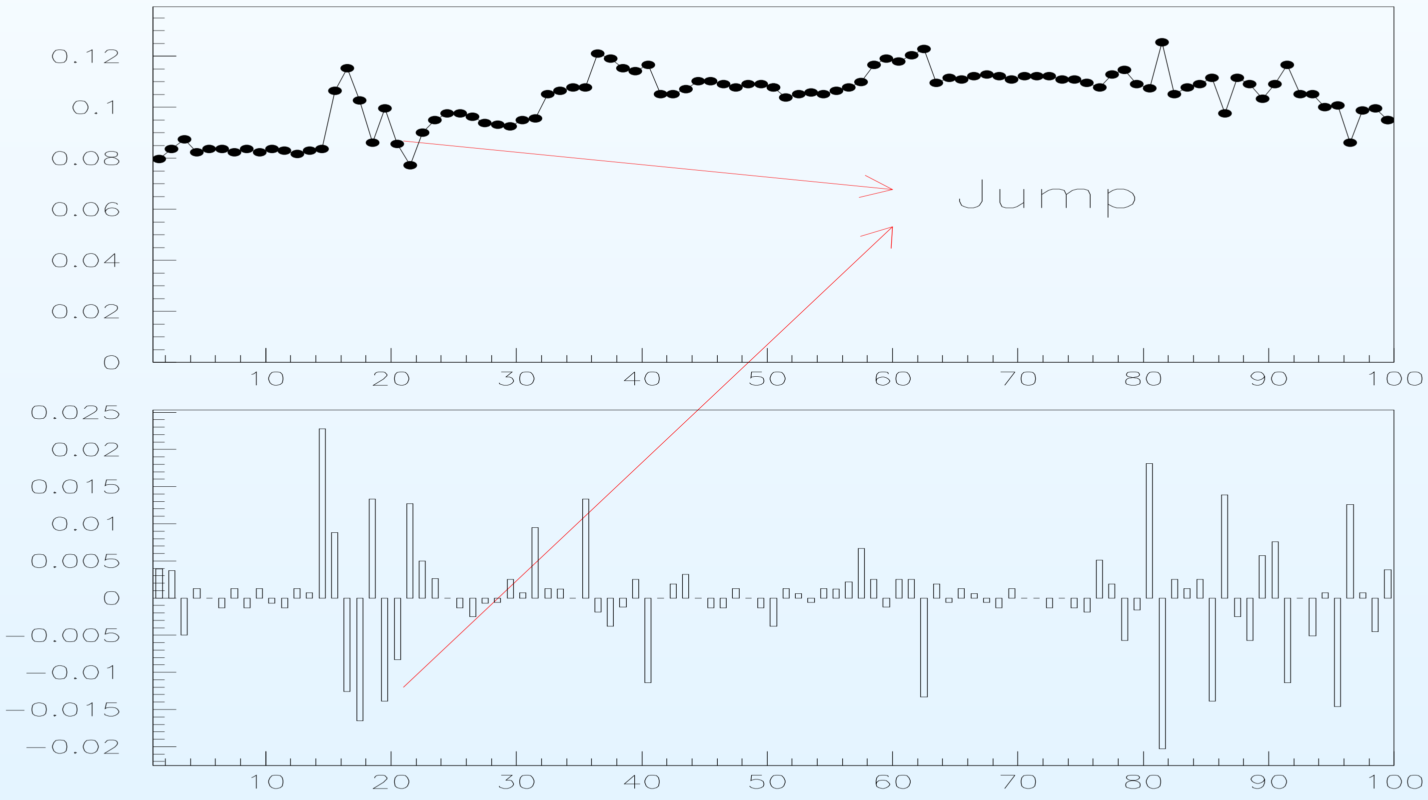
- Conclusions

The econometric problem

When we observe variations in the level of any state variable, are these variations continouous or discontinuous?



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- Commodity prices suffer abrupt spikes due to shortages.
- Electricity prices are highly inelastic to demand, thus spikes are very frequent.

The literature: parametric models

To disentangle jumps from variation we can use:

- **Parametric models**

Examples are Das (2002), Pan (2002), Eraker, Johannes and Polson (2003), Andersen, Benzoni and Lund (2003), Chernov et al. (2003), Piazzesi (2005).

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Test the significance of dJ_t, for example in:

$$\begin{aligned} dX_t &= \mu\, dt + \sqrt{v_t} dW_{x,t} + dJ_t, \\ d \log v_t &= (\alpha - \beta \log v_t)\, dt + \eta dW_{v,t}, \end{aligned} \tag{2}$$

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Ait-Sahalia (2002,2004) directly exploits the properties of likelihood function.

Power variation

Comes from the recent contributions of Barndorff-Nielsen and Shephard, (2004) and Woerner (2005).

- The realized power variation of order γ is defined as:

$$PV_\delta(X)_t^\gamma = \delta^{1-\gamma/2} \sum_{j=1}^{[t/\delta]} |\Delta_j X|^\gamma$$

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$$p - \lim_{\delta \to 0} \mu_\gamma^{-1} PV_\delta(X)_t^\gamma = \begin{cases} \int_0^t \sigma_s^\gamma ds & if\ \gamma < 2 \\ [X]_t & if\ \gamma = 2 \\ +\infty & if\ \gamma > 2 \end{cases}$$

Bipower variation

We define *realized bipower variation* of order $[\gamma_1, \gamma_2]$ as:

$$BPV_\delta(X)_t^{[\gamma_1,\gamma_2]} = \delta^{1-\frac{1}{2}(\gamma_1+\gamma_2)} \sum_{j=2}^{[t/\delta]} |\Delta_{j-1}X|^{\gamma_1} \cdot |\Delta_j X|^{\gamma_2}$$

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$$p - \lim_{\delta\to 0} \mu_{\gamma_1}^{-1} \mu_{\gamma_2}^{-1} BPV_\delta(X)_t^{[\gamma_1,\gamma_2]} =$$

$$= \begin{cases} \int_0^t \sigma_s^{\gamma_1+\gamma_2} ds & if\ max(\gamma_1,\gamma_2) < 2 \\ [X]_t & if\ max(\gamma_1,\gamma_2) = 2 \\ +\infty & if\ max(\gamma_1,\gamma_2) > 2 \end{cases}$$

Standard bipower variation

The $[1,1]$-order bipower variation, when it exists, is defined as:

$$BPV(X)_t^{[1,1]} = p-\lim_{\delta\to 0}\sum_{j=2}^{[t/\delta]} |\Delta_{j-1}X|\cdot|\Delta_j X|$$

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In the case $\mu = 0$ and σ independent from W_t, we have:

$$BPV(X)_t^{[1,1]} = \mu_1 \int_0^t \sigma_s^2 ds = \mu_1^2 [X^c]_t$$

where

$$\mu_1 = \frac{\sqrt{2}}{\sqrt{\pi}}$$

This result can be extended to the case $\mu \neq 0$

Estimation based on sample moments

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- Suppose $\lambda = \lambda(X_t)$ and the jump sizes $c_i = C$ are all identically distributed. Then:

$$\lim_{\delta \to 0} \frac{1}{\delta} \mathbb{E}_t \left[X_{t+\delta} - X_t \right] = \mu(X_t)$$

$$\lim_{\delta \to 0} \frac{1}{\delta} \mathbb{E}_t \left[(X_{t+\delta} - X_t)^2 \right] = \sigma(X_t) + \lambda(X_t) \mathbb{E} \left[C^2 \right]$$

$$\lim_{\delta \to 0} \frac{1}{\delta} \mathbb{E}_t \left[(X_{t+\delta} - X_t)^k \right] = \lambda(X_t) \mathbb{E} \left[C^k \right]$$

Threshold estimation

Mancini (2003,2007) disentangles diffusion from jumps using the *modulus of continuity* of the Brownian motion:

$$r(\delta) = \sqrt{2\delta \log \frac{1}{\delta}}$$

which has the following property, as established by Lévy:

$$\mathcal{P}\left[\limsup_{\delta\to 0} \frac{\max_{|t-s|\leq\delta} |W(t) - W(s)|}{r(\delta)} = 1\right] = 1$$

It measures the *speed* at which the Brownian motion shrinks to zero.

The intuition

When $\delta \to 0$, diffusive variations go to zero, while jumps do not.

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Thus, we can identify the jumps as those variations which are larger than a suitable *threshold* $\vartheta(\delta)$ which goes to zero, as $\delta \longrightarrow 0$, slower than $r(\delta)$.

The theorem (Mancini, 2007)

Suppose

$$X = Y + J$$

Where Y is a Brownian martingale plus drift and J is a jump process with counting process N with $E[N_T] < \infty$ and time horizon $T < \infty$.

If $\vartheta(\delta)$ is a real deterministic function such that

$$\lim_{\delta \to 0} \vartheta(\delta) = 0 \quad \text{and} \quad \lim_{\delta \to 0} \frac{\delta \log \frac{1}{\delta}}{\vartheta(\delta)} = 0$$

then for P-almost all ω, $\exists \bar{\delta}(\omega)$ such that $\forall \delta < \bar{\delta}(\omega)$ we have

$$\forall i = 1, \ldots, n, \quad I_{\{\Delta N = 0\}}(\omega) = I_{\{(\Delta X)^2 \leq \vartheta(\delta)\}}(\omega).$$

Threshold power and multipower variation

Power variation can be generalized with the following definition of threshold power variation, for $\gamma \geq 1$:

$$tPV_\delta(X)_t^{[\gamma]} = \delta^{1-\frac{1}{2}\gamma} \sum_{j=1}^{[t/\delta]} |\Delta_j X|^\gamma I_{\{|\Delta_j X|^2 \leq \vartheta_j(\delta)\}} \longrightarrow \mu_\gamma \int_0^t \sigma_s^\gamma ds \qquad (4)$$

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Limit theorems in law for tPV in the case $\gamma = 2$ are established in Mancini (2007).

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Limit theorems in law for tPV in the case $\gamma = 2$ are established in Mancini (2007).

We define the (realized) threshold multipower variation as:

$$TMPV_\delta(X)_t^{[\gamma_1,\ldots,\gamma_M]} = \delta^{1-\frac{1}{2}(\gamma_1+\ldots+\gamma_M)} \sum_{j=M}^{[t/\delta]} \prod_{k=1}^{M} |\Delta_{j-k+1} X|^{\gamma_k} I_{\{|\Delta_{j-k+1} X|^2 \leq \vartheta_{j-k+1}(\delta)\}} \qquad (9)$$

Threshold multipower variation asymptotics

As $\delta \longrightarrow 0$ we have (Pirino and Renò, 2007)

1. if $\max(\gamma_1, \ldots, \gamma_M) < 2$ we have:

$$TMPV_\delta(X)_t^{[\gamma_1,\ldots,\gamma_M]} \longrightarrow \left(\prod_{k=1}^{M} \mu_{\gamma_k}\right) \int_0^t \sigma_s^{\gamma_1+\ldots+\gamma_M} ds \qquad (10)$$

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2. if $\max(\gamma_1, \ldots, \gamma_M) < 1$ we have:

$$\delta^{-\frac{1}{2}} \left(TMPV_\delta(X)_t - \int_0^t \sigma_s^{\gamma_1+\ldots+\gamma_M} ds \right) \longrightarrow c_\gamma \int_0^t \sigma_s^{\gamma_1+\ldots+\gamma_M} dW_s' \qquad (13)$$

   where the above convergence is in law.

The model

We use the above intuition to estimate univariate models of the kind:

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t + dJ_t, \qquad t \in [0, T],$$

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This is the first estimator of the diffusion coefficient under an infinite activity jump component.

A primer to nonparametric estimation

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Suppose that n observations $X_1, \ldots, X_n$ are drawn from an unknown density $f(x)$, and we want to estimate the density.

Histograms

One popular way to estimate the density is the histogram.
The estimate through an histogram can be written:

$$\hat{f}(x) = \frac{1}{nh} \sum_{i=1}^{n} I_{\{|X_i - x| < h/2\}}$$

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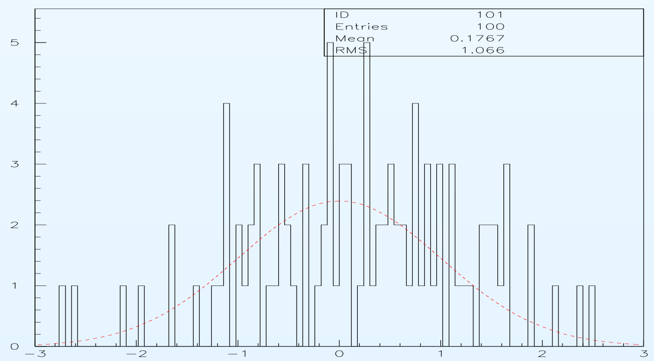

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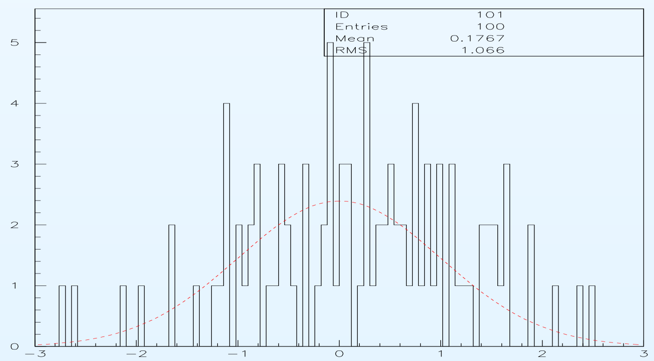

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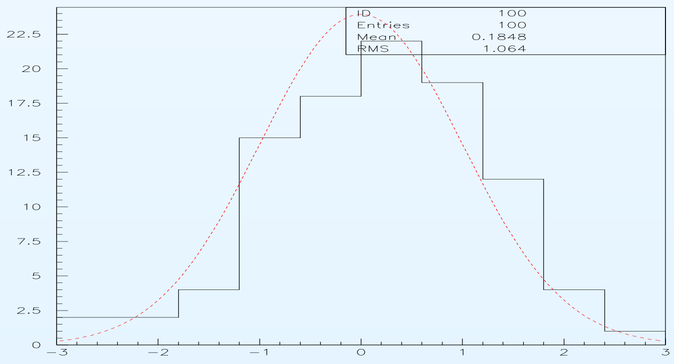

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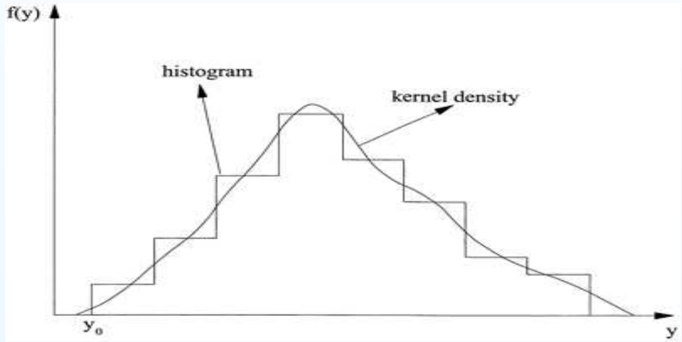
$$\hat{f}(x) = \frac{F(x+\frac{h}{2}) - F(x-\frac{h}{2})}{h} = \frac{1}{nh}\sum_{i=1}^{n} I_{\{|X_i - x| < h/2\}}$$

Kernel smoothing

Histograms are not smooth by definition. We can obtain a smoother estimate replacing the indicator function with a continuous function, the *kernel* $K(\cdot)$:

$$\hat{f}(x) = \frac{1}{nh}\sum_{i=1}^{n} K\left(\frac{X_i - x}{h}\right)$$



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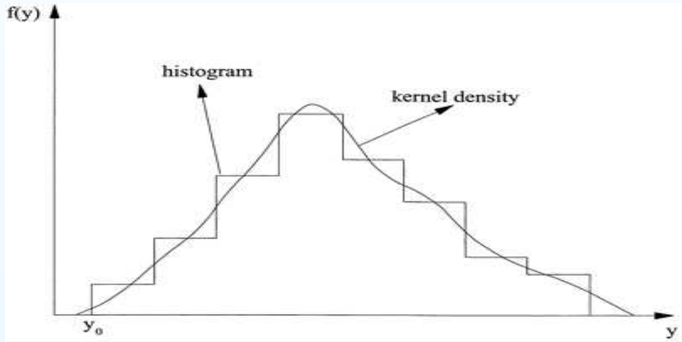

A kernel is a function $K : \mathbb{R} \in \mathbb{R}^{+}$ such that

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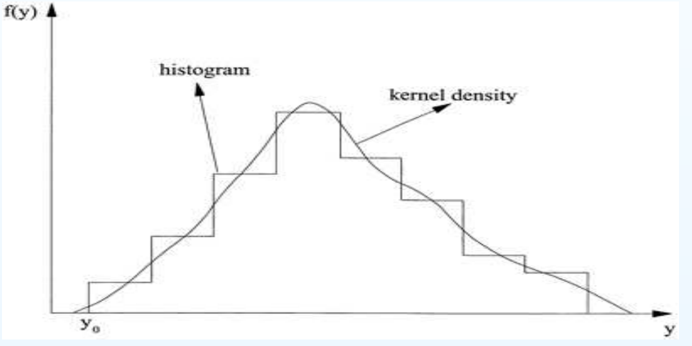

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A bandwidth h is a sequence of positive real number such that

$$\lim_{n\to\infty} h = 0, \quad \lim_{n\to\infty} nh = +\infty$$

Smooth histograms

One example with $K(s) = \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{2}}$, the Gaussian kernel.

The choice of h has an impact on the estimate:

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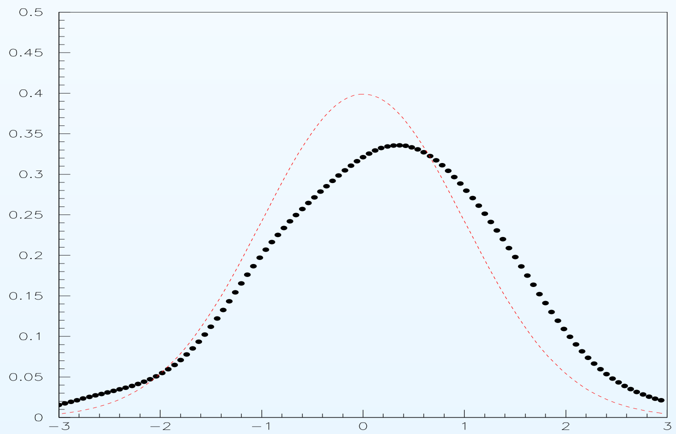

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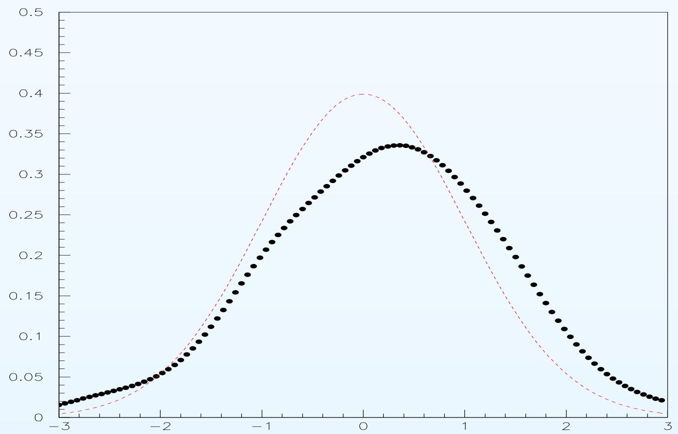
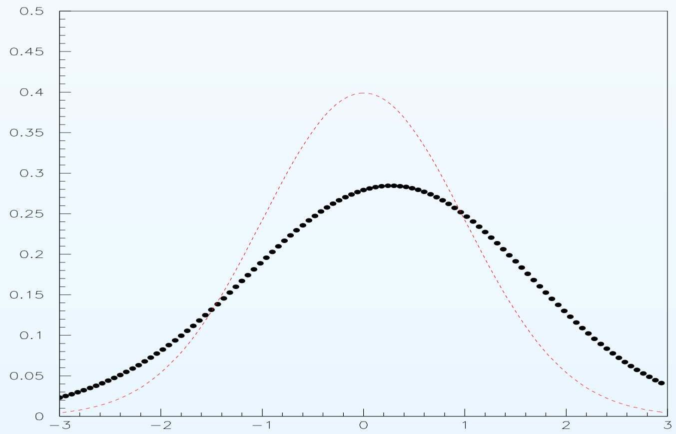
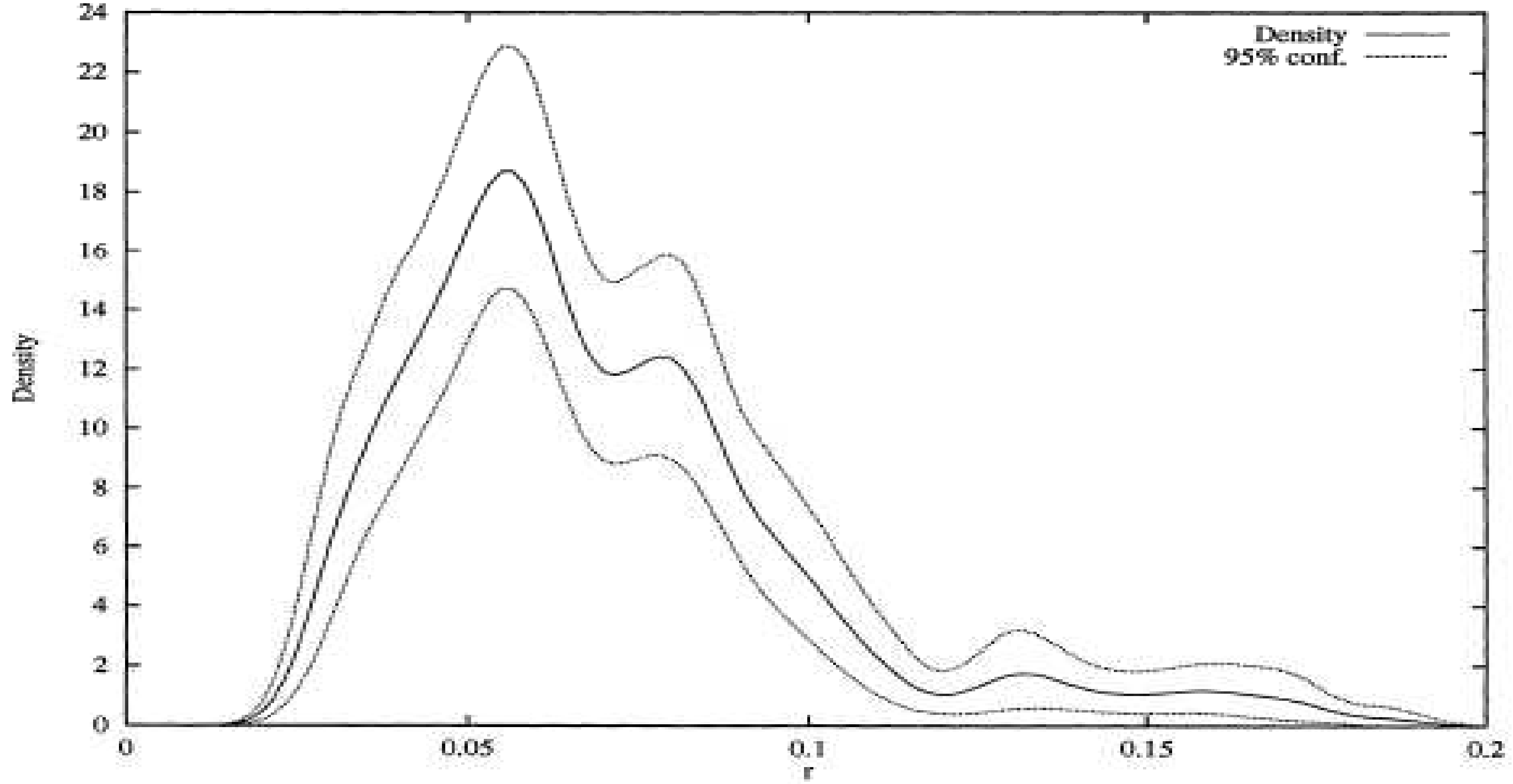


oversmoothing



Interest rate density

Font: Stanton (1997)



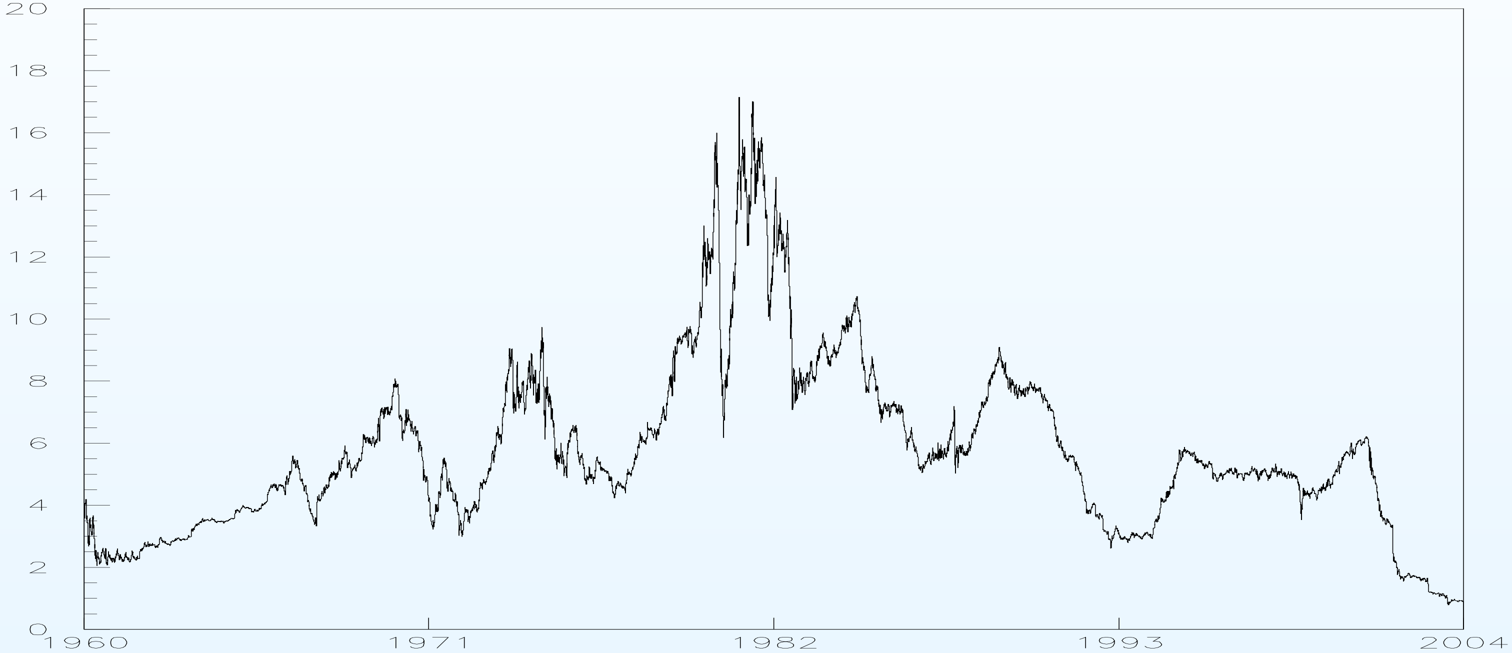
Nonparametric estimation: the continuous case

A popular estimator is that proposed by Florens-Zmirou (1993), and subsequently by Stanton (1997) and Jiang and Knight (1998).
The estimator is very intuitive:

$$\hat{\sigma}^2(r) = \frac{n \sum_{t=1}^{n-1} K\left(\frac{r-\hat{r}_t}{h}\right) (\hat{r}_{t+1} - \hat{r}_t)^2}{T \sum_{t=1}^{n} K\left(\frac{r-\hat{r}_t}{h}\right)}.$$

Asymptotic properties are fully assessed in Florens-Zmirou (1993).
A generalization is given by Bandi and Phillips (2003).

The intuition behind non-parametric estimation



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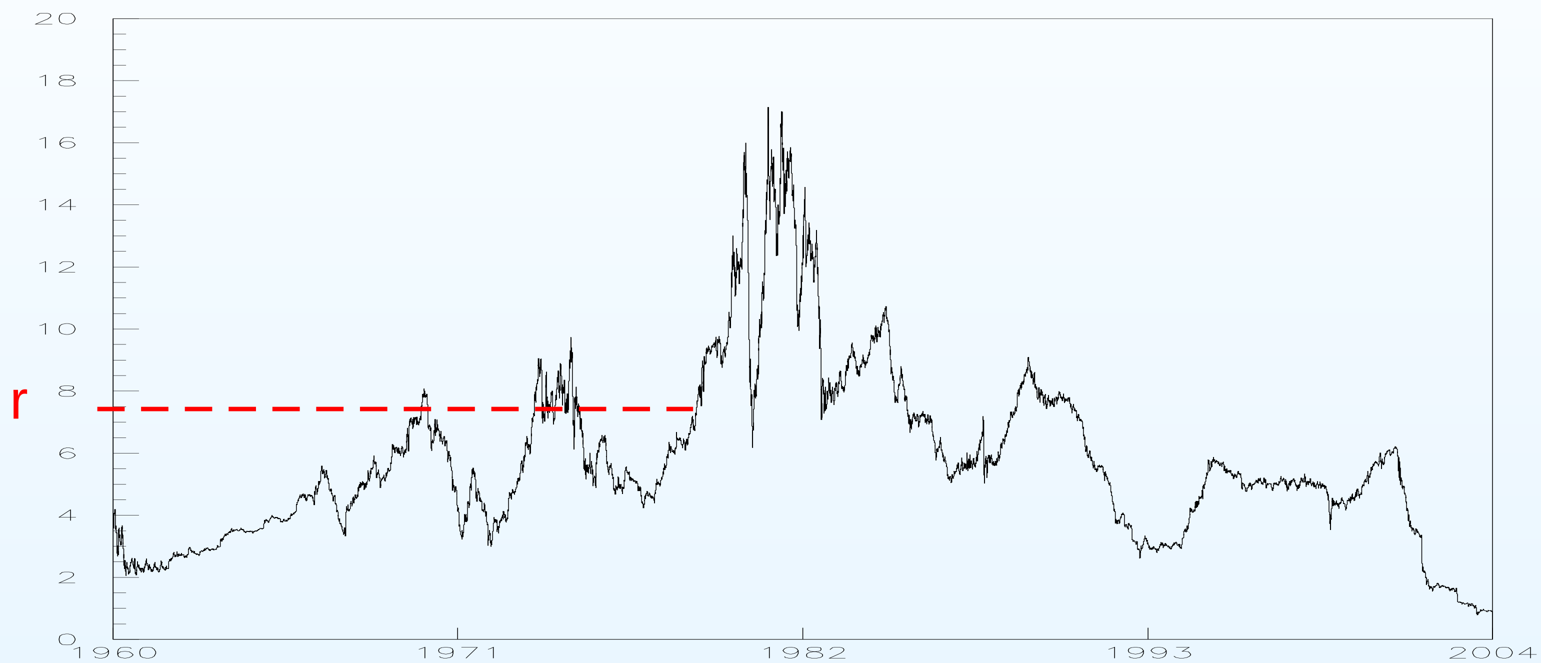
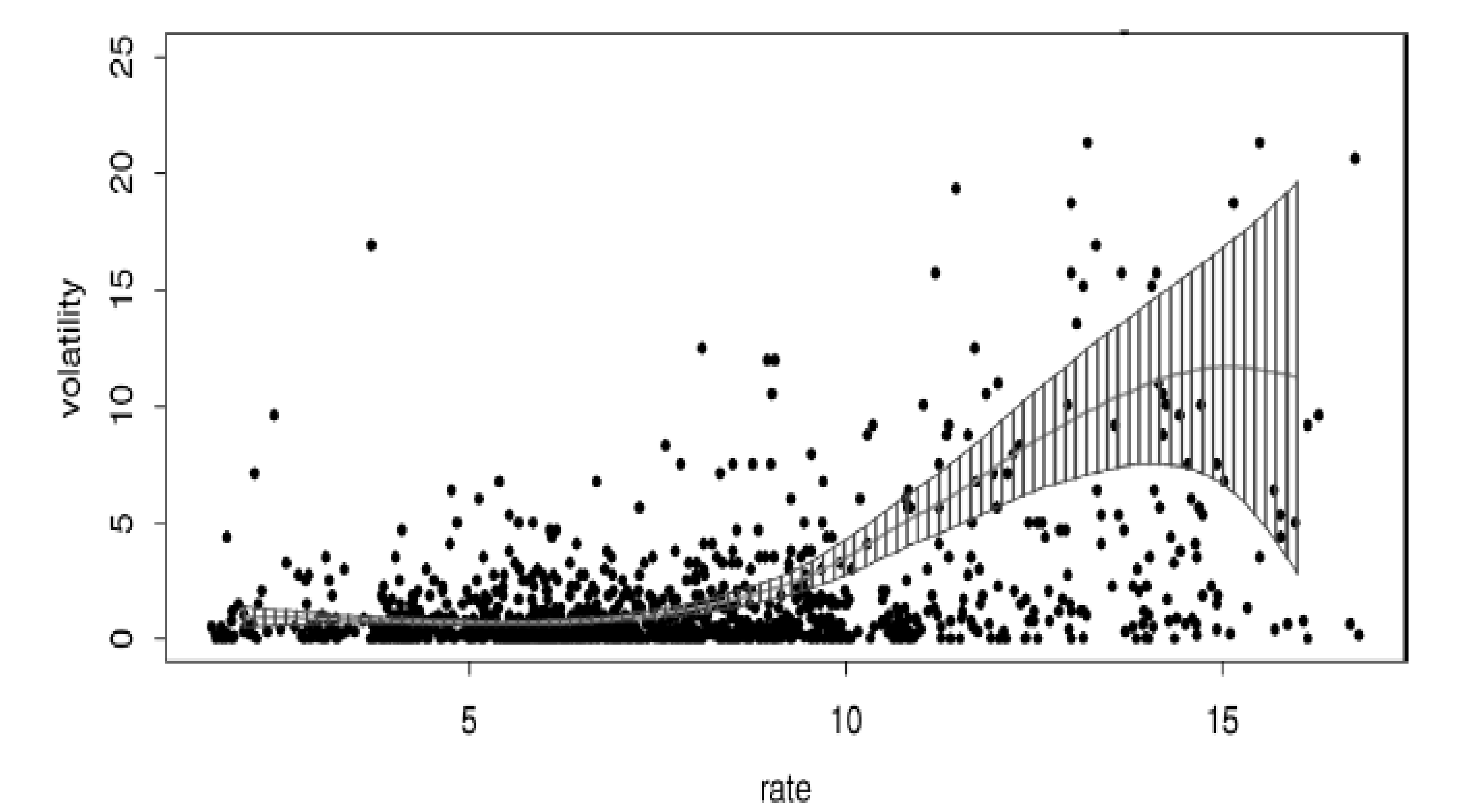


The intuition behind non-parametric estimation

Estimate the quadratic variation here!

The diffusion coefficient estimator

Font: Fan (2003)



Bandi-Phillips estimators

Bandi and Phillips (2003) propose an alternative estimator which is very close to that of Florens-Zmirou (1993):

$$\hat{\sigma}^2(r) = \frac{n \sum_{i=1}^{n} K\left(\frac{r-\hat{r}_i}{h}\right)\left(\frac{1}{m_i}\sum_{j=0}^{m_i}\left[\hat{r}_{t_{i,j}+1} - \hat{r}_{t_{i,j}}\right]^2\right)}{T \sum_{i=1}^{n} K\left(\frac{r-\hat{r}_i}{h}\right)}$$

where $t_{i,j}$ is a subset of indexes such that

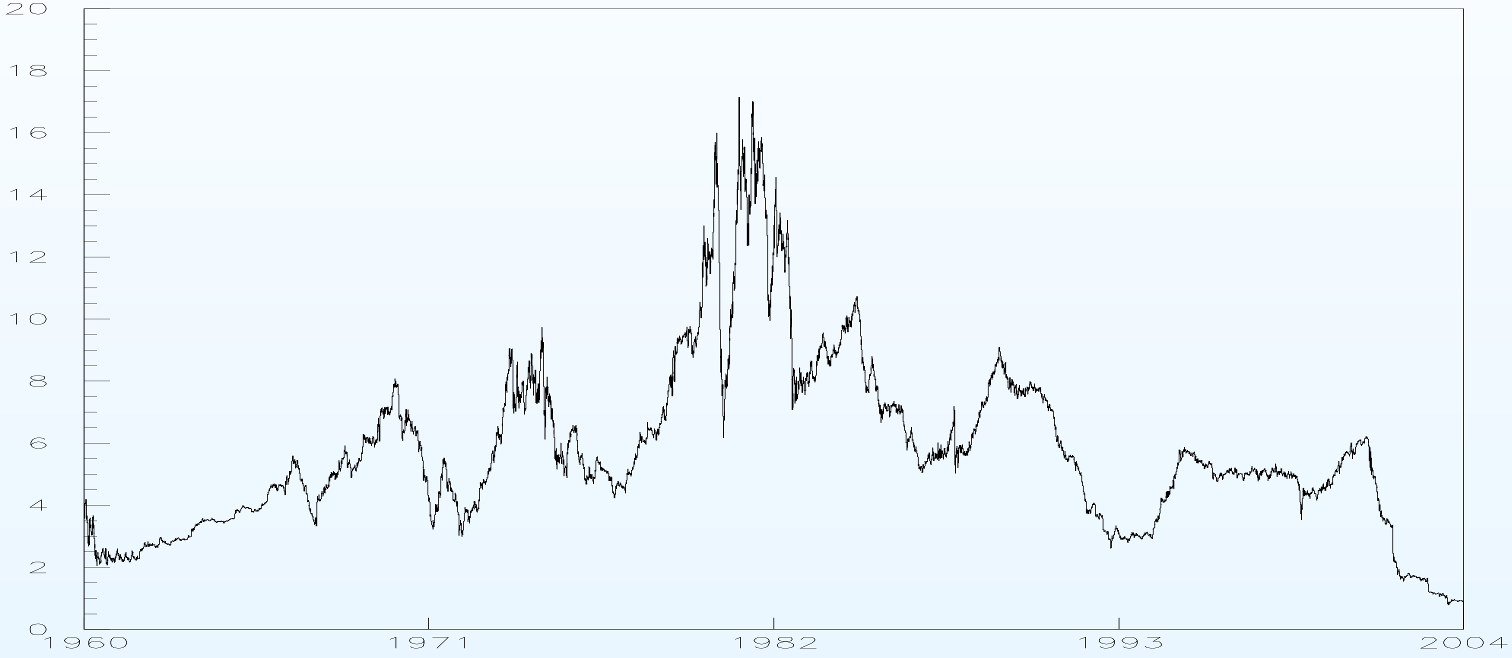
$$t_{i,0} = \inf\{t \geq 0 : |\hat{r}_t - \hat{r}_i| \leq \varepsilon_s\},$$

and

$$t_{i,j+1} = \inf\left\{t \geq t_{i,j} + 1 : |\hat{r}_t - \hat{r}_i| \leq \varepsilon_s\right\},$$

m_i is the number of times that $|\hat{r}_t - \hat{r}_i| \leq \varepsilon_s$ and ε_s is a parameter.

The intuition of Bandi-Phillips



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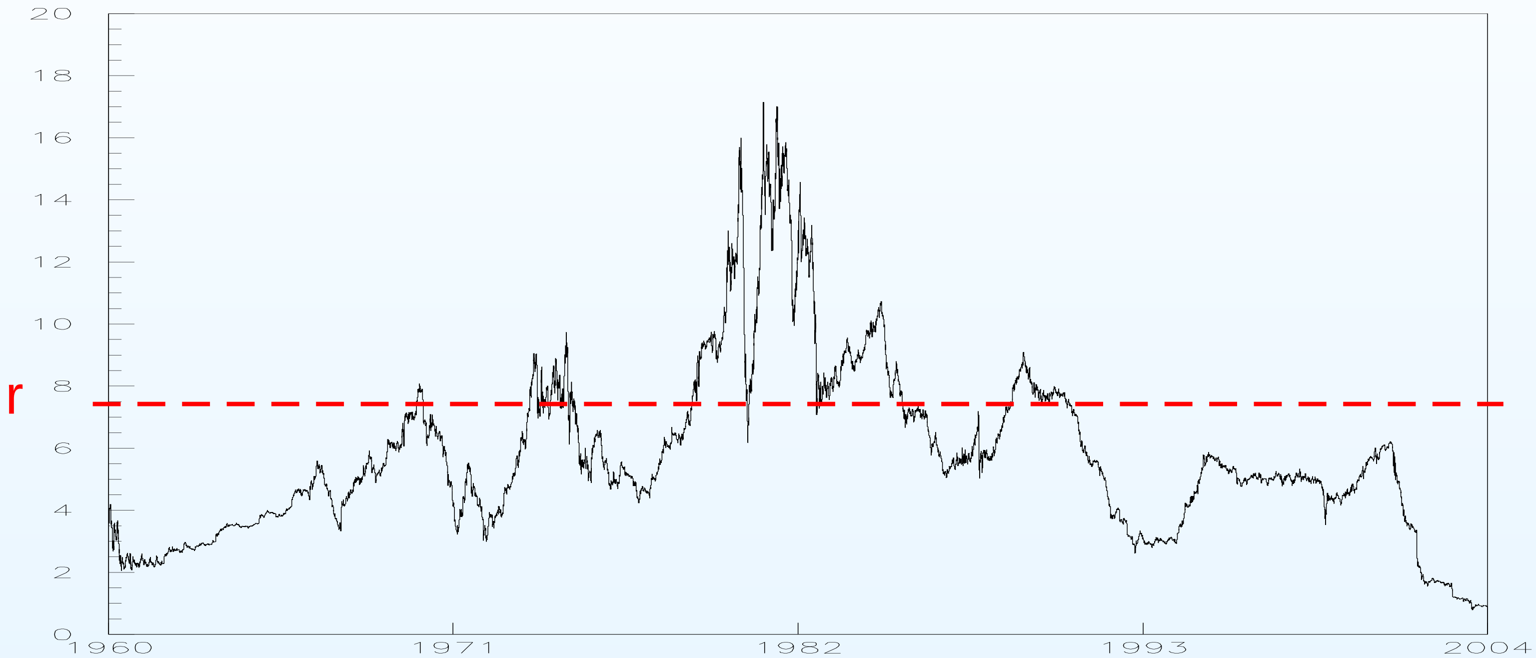

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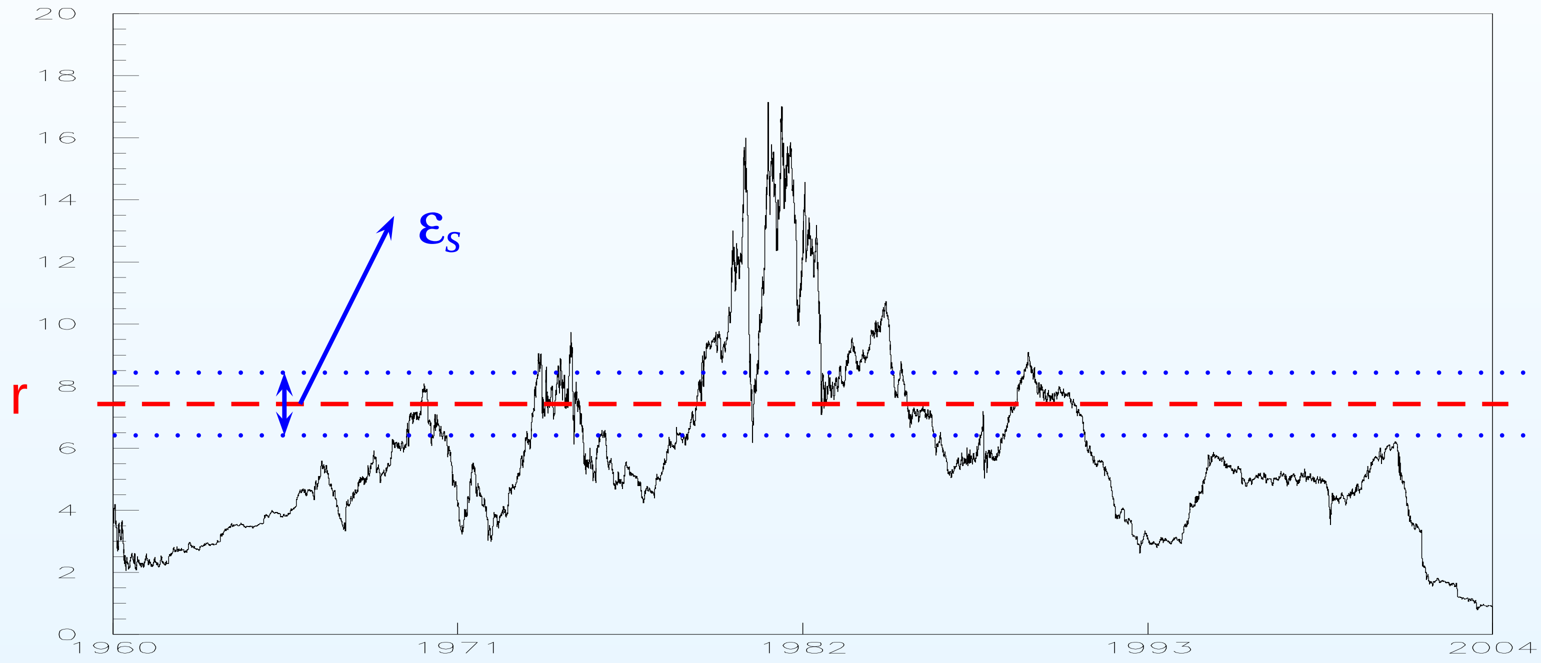

Estimate the quadratic variation in the strip

Estimation of the jump process

We first get (in the case of finite activity) an estimate of the whole jump process using:

$$\hat{J}_{1,t} = \sum_{\{i: t_i \le t\}} \hat{\gamma}_{\tau^{(i)}},$$

where

$$\hat{\gamma}_{\tau^{(i)}} \doteq (X_{t_{i+1}} - X_{t_i}) \cdot I_{\{(X_{t_{i+1}} - X_{t_i})^2 > \vartheta(\delta)\}}$$

It is remarkable that we estimate contemporaneously jump times and sizes.

The proposed estimator

Let X be a jump-diffusion process with finite activity, and assume that:

- $P\{\gamma_\ell = 0\} = 0$;
- $t_i = i\delta$ (equally spaced observations)
- as $\delta \to 0$ both the threshold function $\vartheta(\delta)$ and $\frac{\delta \ln \frac{1}{\delta}}{\vartheta(\delta)}$ tend to zero;
- $nh^4 \to 0$ as $n \to \infty$ and $\exists \beta > 1 : nh^\beta \to \infty$.

Then for all x visited by X

$$\hat{\sigma}^2_{n,h}(x) = \frac{n \sum_{i=1}^{n} K\left(\frac{X_{t_i} - \hat{J}_{1,t_i} - x}{h}\right) (X_{t_{i+1}} - X_{t_i})^2 I_{\{(X_{t_{i+1}} - X_{t_i})^2 \leq \vartheta(\delta)\}}}{T \sum_{i=1}^{n} K\left(\frac{X_{t_i} - \hat{J}_{1,t_i} - x}{h}\right)} \to_P \sigma^2(x)$$

Threshold setting

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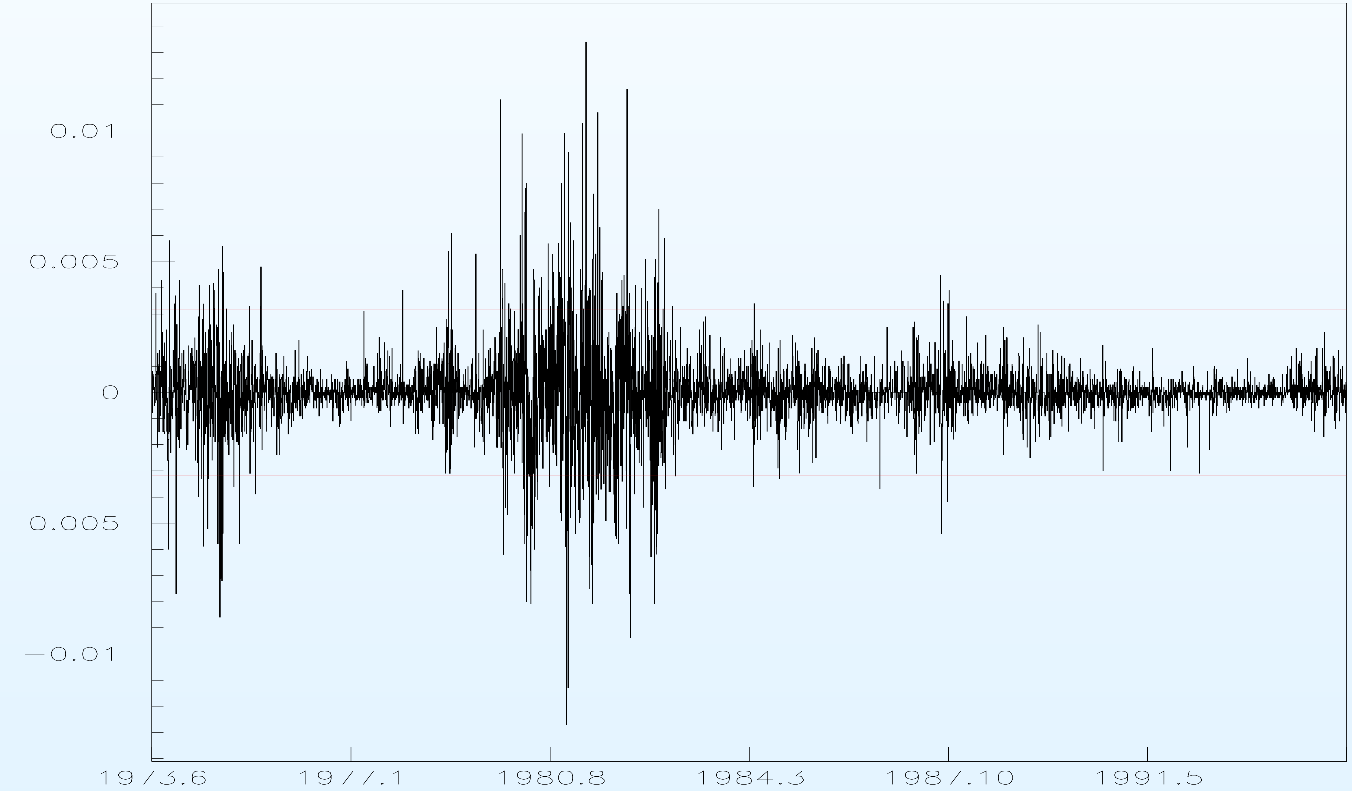
If ϑ is too small $\longrightarrow$ many variations will be detected as jumps.

If ϑ is too large $\longrightarrow$ many jumps will not be detected.

An auxiliary model for the threshold

Moreover, we need a non-constant threshold to take into account volatility persistence.



An auxiliary model for the threshold

Moreover, we need a non-constant threshold to take into account volatility persistence.

We use a GARCH(1,1) model. Even if misspecified, it asymptotically provides the optimal volatility forecast (Nelson and Foster, 1994).

We then fit:

$$r_t - r_{t-1} = \bar{r} + \sqrt{h_t} \cdot \varepsilon_t$$
$$h_t = \omega + \alpha(r_{t-1} - r_{t-2})^2 + \beta h_{t-1}$$

then we set:

$$\vartheta(t) = c \cdot h_t$$

with $c = 9$ (three standard deviations).

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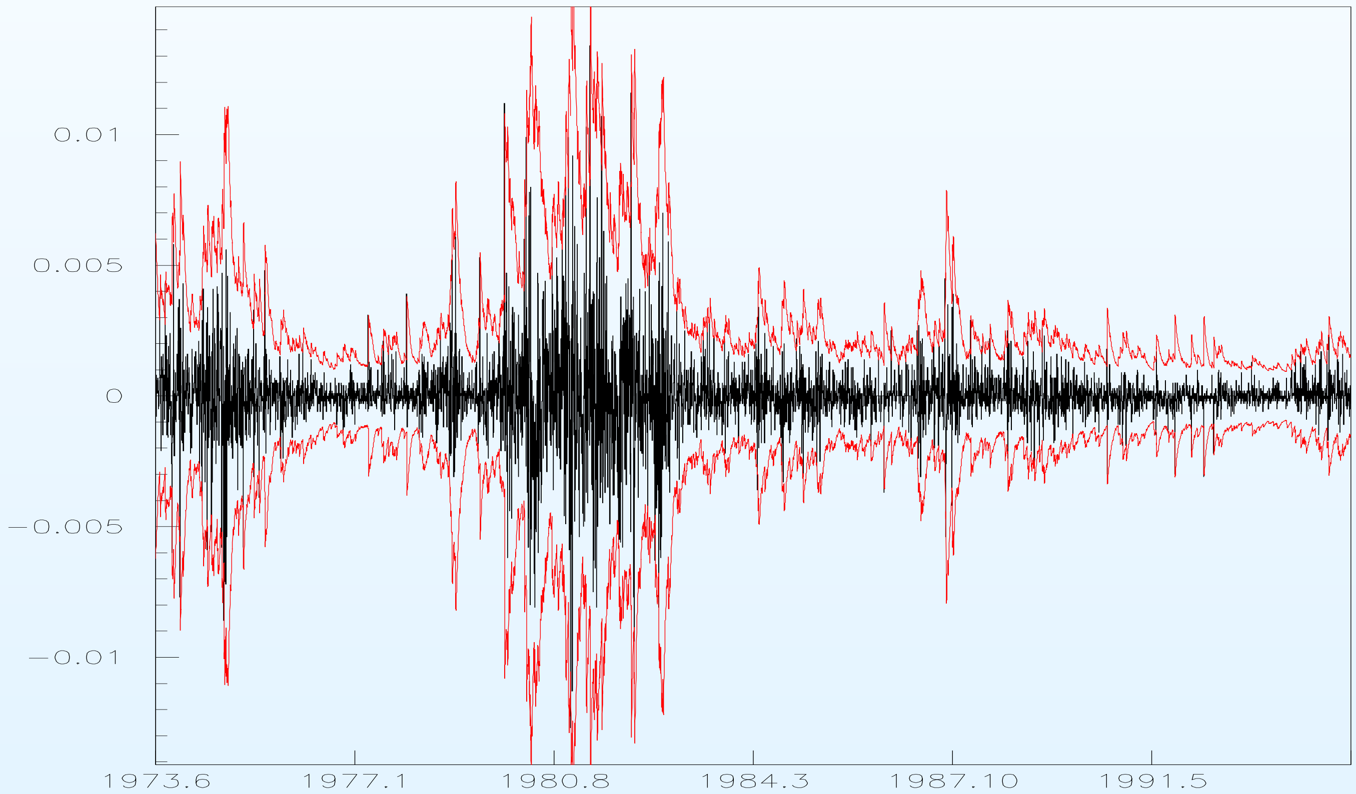


The simulated model

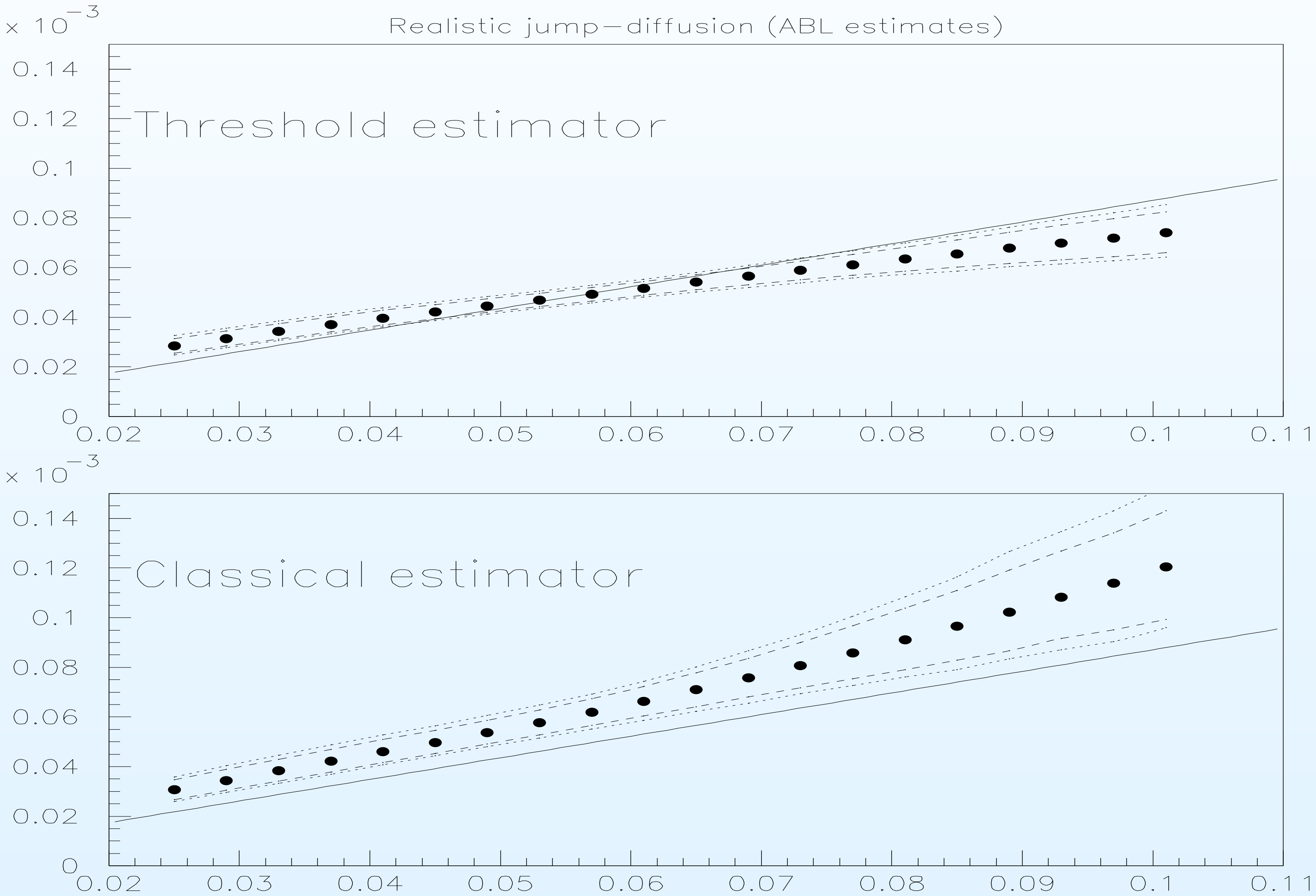
$$
\begin{aligned}
dr_t &= \kappa_1(b_t - r_t)dt - \bar{\kappa}\lambda r_t dt + \gamma\sqrt{r_t}dW_{1,t} + (e^{Z_t} - 1)r_t dN_t \\
db_t &= \kappa_3(\theta - b_t)dt + \eta_2\sqrt{b_t}dW_{2,t}, \\
Z_t &\simeq \mathcal{N}(\mu_J, \sigma_J^2), \\
\bar{\kappa} &= \mathbb{E}\left[e^{Z_t} - 1\right] = e^{\left(\mu_J + \frac{1}{2}\sigma_J^2\right)},
\end{aligned}
$$

This is a modified version of the models estimated by Andersen, Benzoni and Lund (2004) on a time series of 3-months Treasury Bills annualized rates.

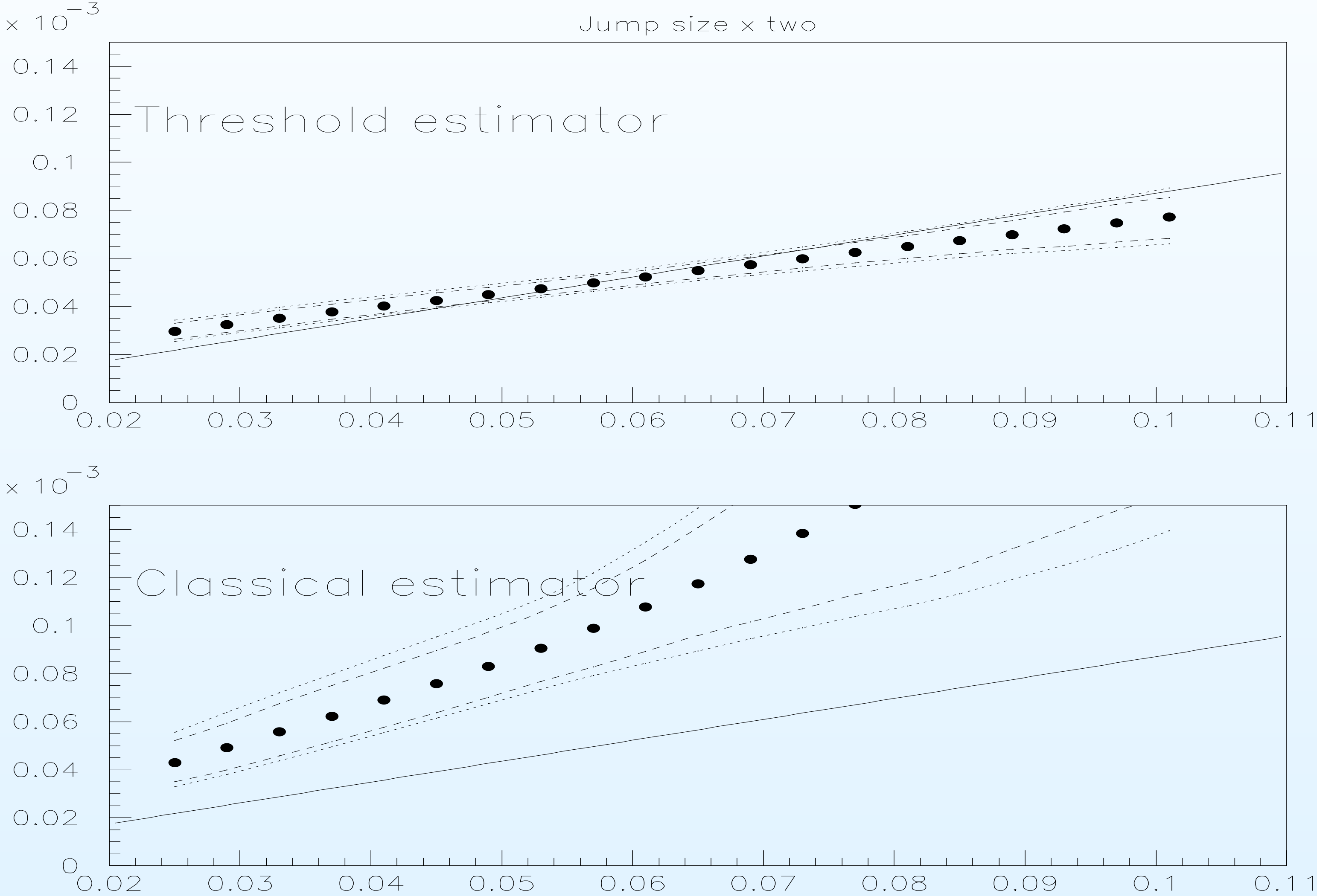
We use their estimates of the $SV_1J - SD$ model.

The starting values r_0 and b_0 are sampled from the unconditional distribution.

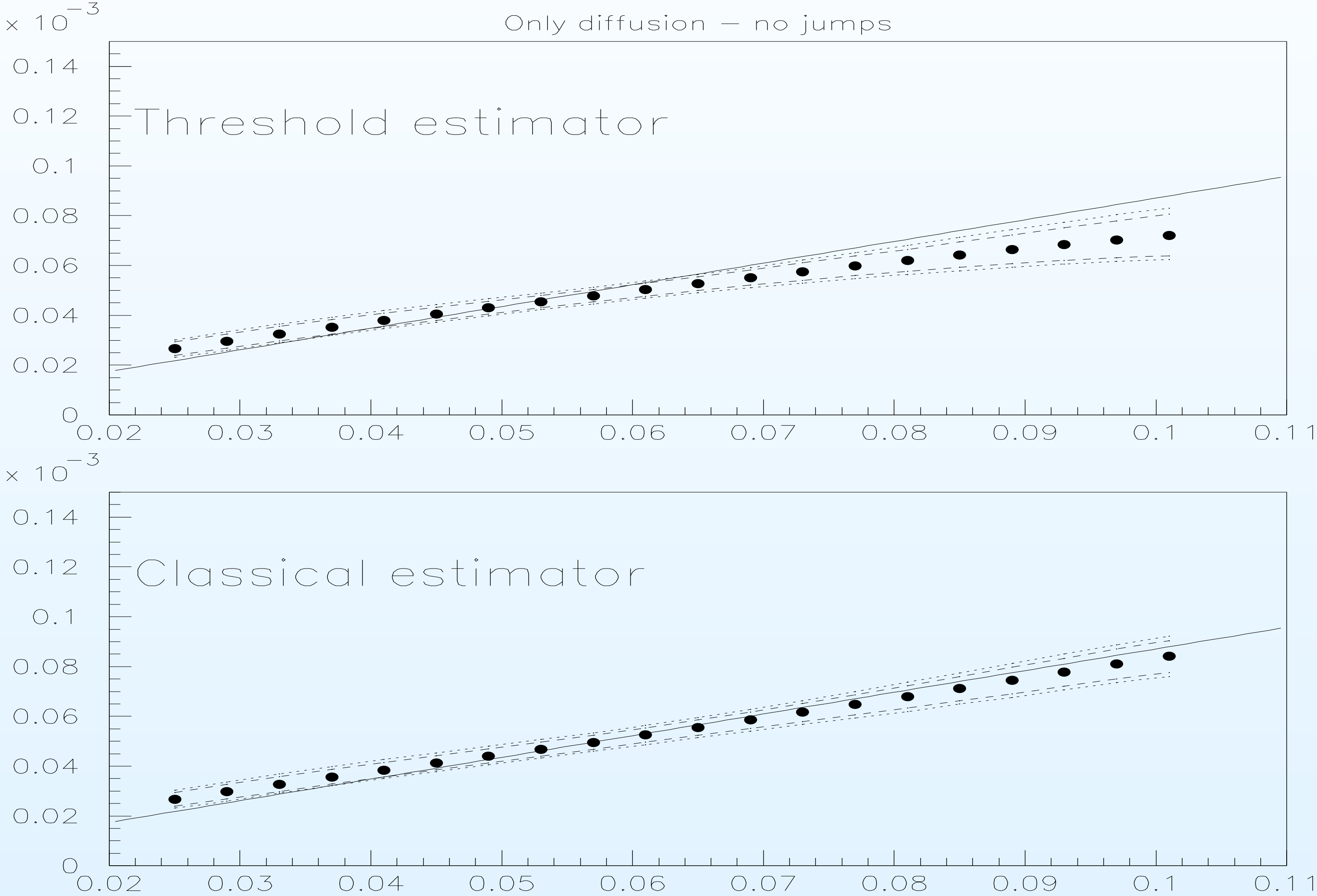
Simulations - realistic jump sizes



Simulations - doubled jump sizes



Simulations - no jumps



Estimating the drift function

When J is a doubly stochastic Poisson process it is possible to estimate the drift and the jump intensity functions by letting $T \longrightarrow \infty$ and $T/n \longrightarrow 0$.

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The estimator for the drift is:

$$\hat{\mu}_n(x) = \frac{n \sum_{i=1}^{n} K\left(\frac{X_{t_{i-1}} - \hat{J}_{1,t_{i-1}} - x}{h}\right) (X_{t_{i+1}} - X_{t_i}) \cdot I_{\{(X_{t_{i+1}} - X_{t_i})^2 \leq \vartheta(\frac{T}{n})\}}}{T \sum_{i=1}^{n} K\left(\frac{X_{t_{i-1}} - \hat{J}_{1,t_{i-1}} - x}{h}\right)}$$

Then for each x visited by X we have

$$\hat{\mu}_n(x) \longrightarrow_P \mu(x).$$

Estimation of the intensity function

The estimator for $\lambda(x)$ is:

$$\hat{\lambda}_n(x) = \frac{n \sum_{i=1}^{n} K\left(\frac{X_{t_{i-1}} - x}{h}\right) c_{i,n} I_{\{(X_{t_{i+1}} - X_{t_i})^2 > \vartheta(\delta)\}}}{T \sum_{i=1}^{n} K\left(\frac{X_{t_{i-1}} - x}{h}\right)}$$

where $\sup_i |1 - c_{i,n}| \to 0$ when $n \to \infty$.

Then, for each x visited by X,

$$\hat{\lambda}_n(x) \to_P \lambda(x).$$

A small sample correction

The coefficients c_i can help in recovering an unbiased estimated intensity by making assumptions on the distribution of jumps.

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We can estimate σ_J using, for example, a simple method of moments estimator, fitting:

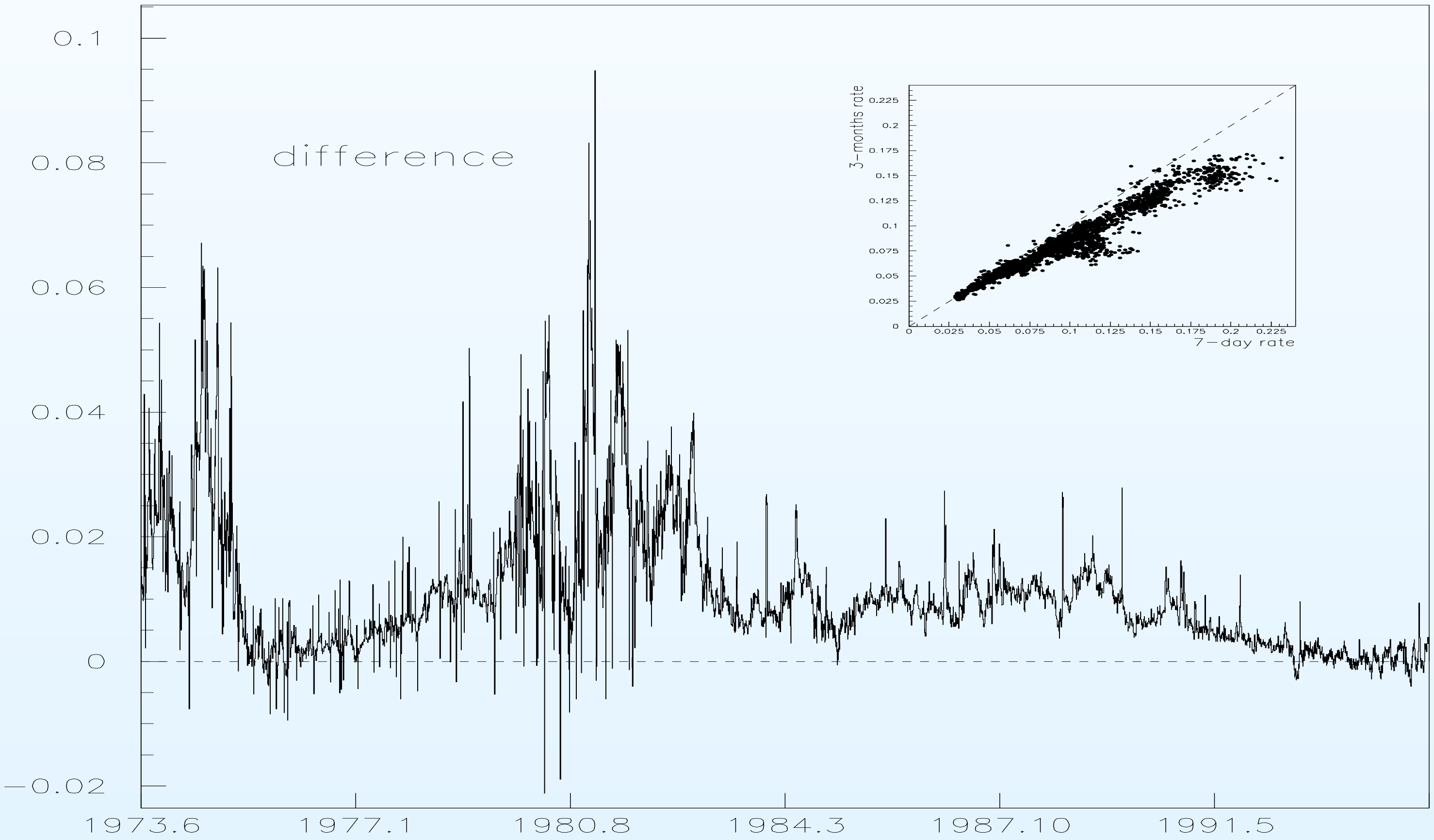
$$m_2(\vartheta) = \sigma_J^2 + \frac{\sigma_J\sqrt{\vartheta}\exp\left(\frac{-\vartheta}{2\sigma_J^2}\right)}{\mathcal{N}(-\sqrt{\vartheta}/\sigma_J)\sqrt{2\pi}} \tag{16}$$

Short rate modelling

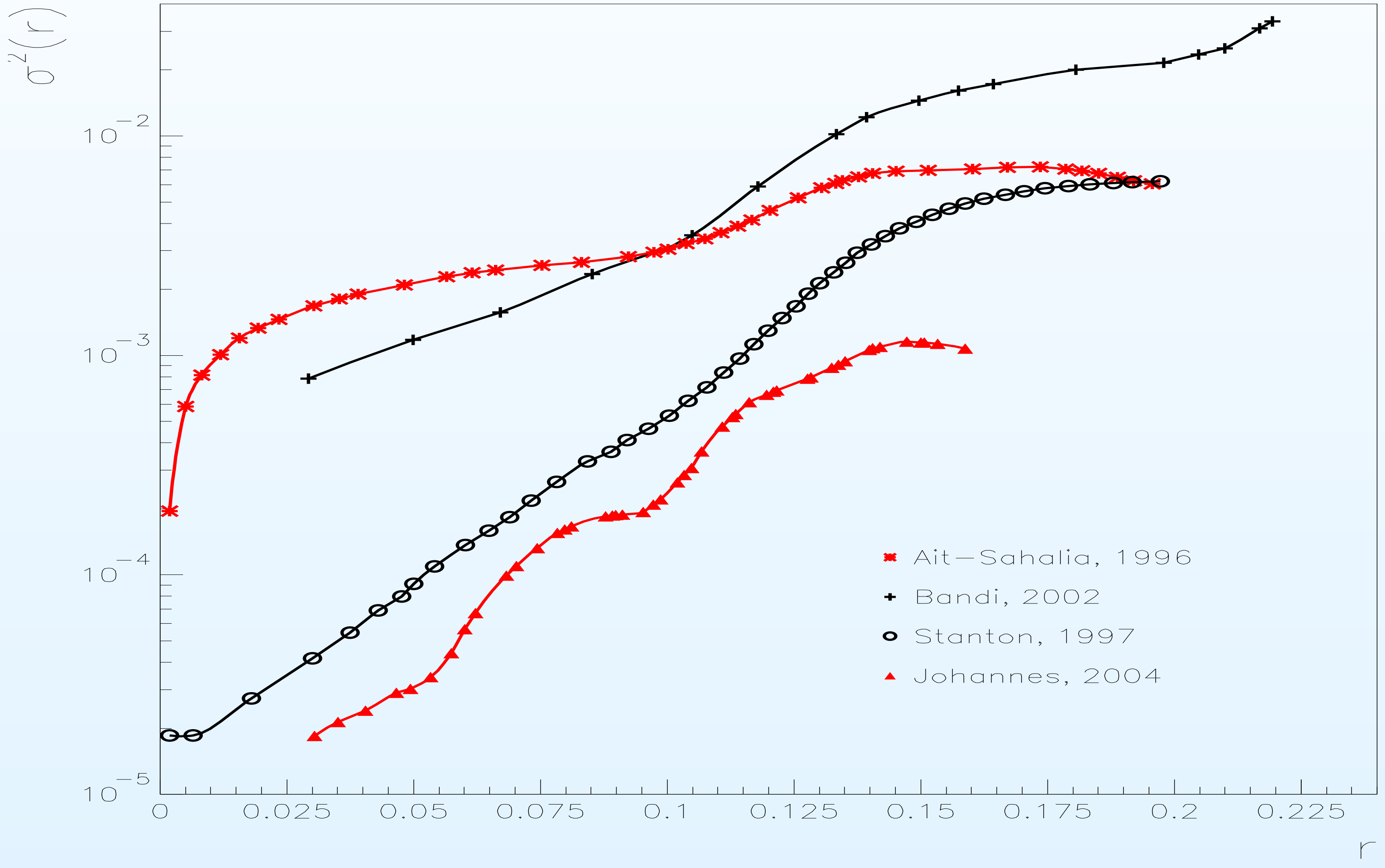
Which is the short rate?

Short rate modelling

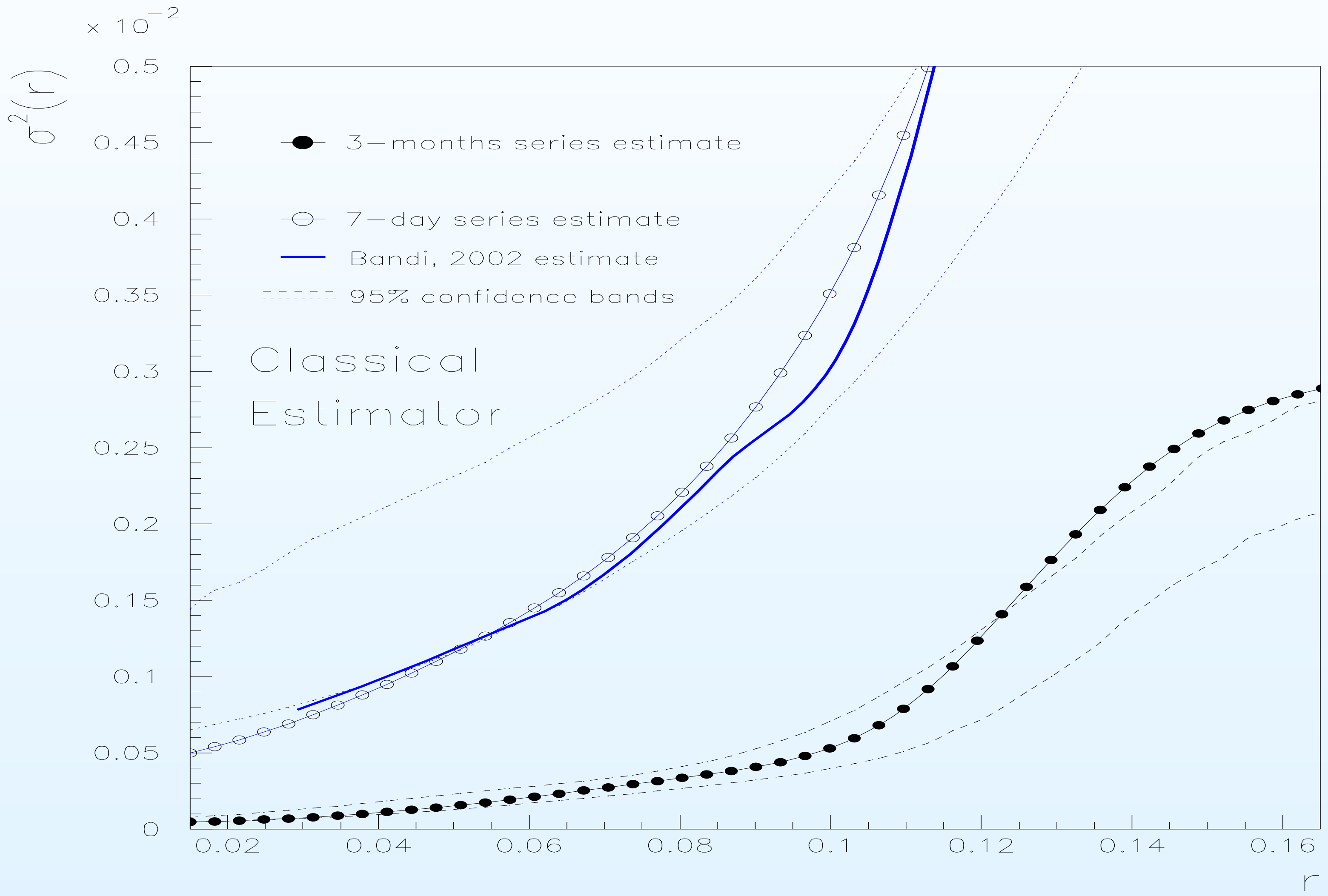
Which is the difference?



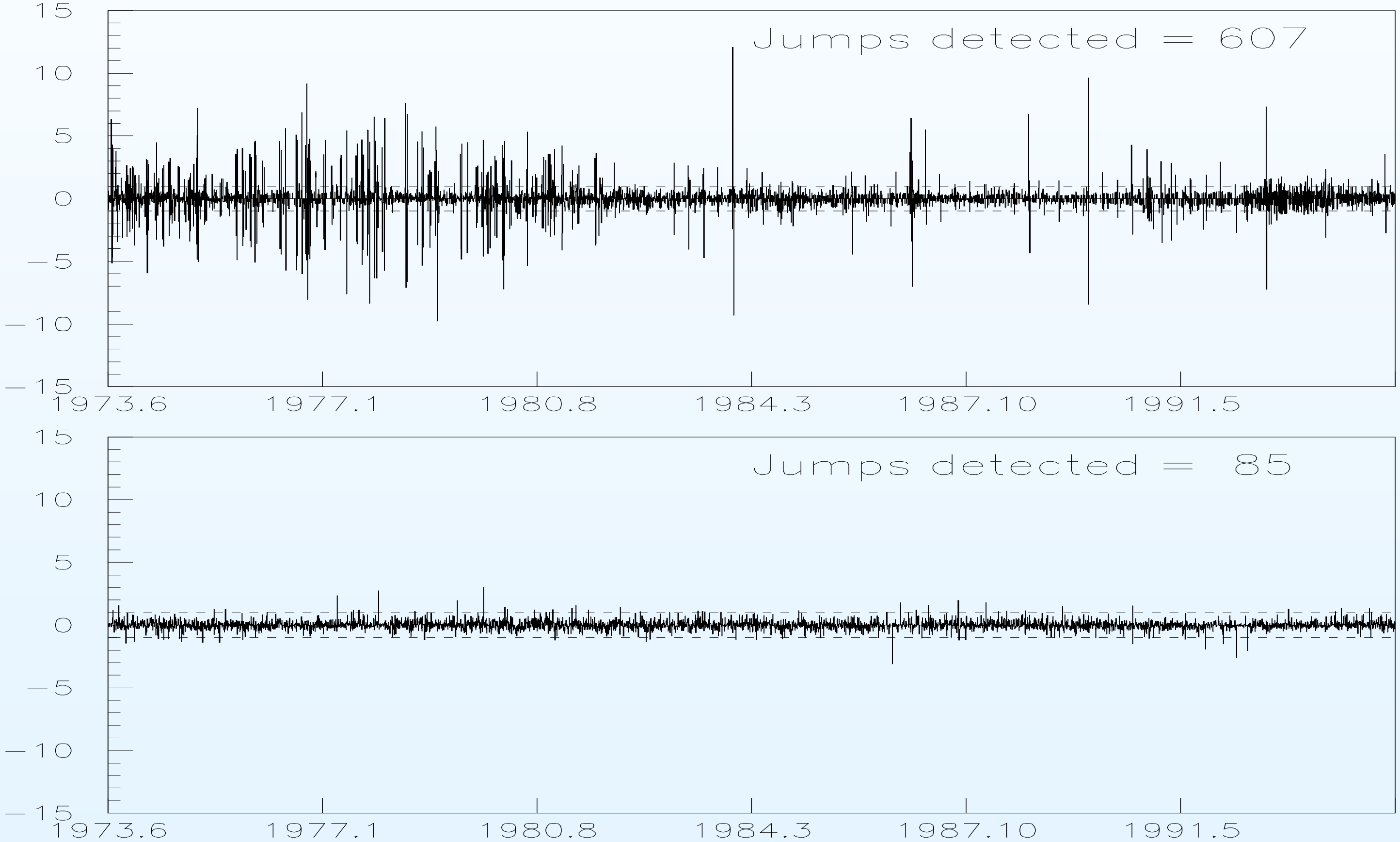
The state of the art



Estimation with no jumps



Jumps detected



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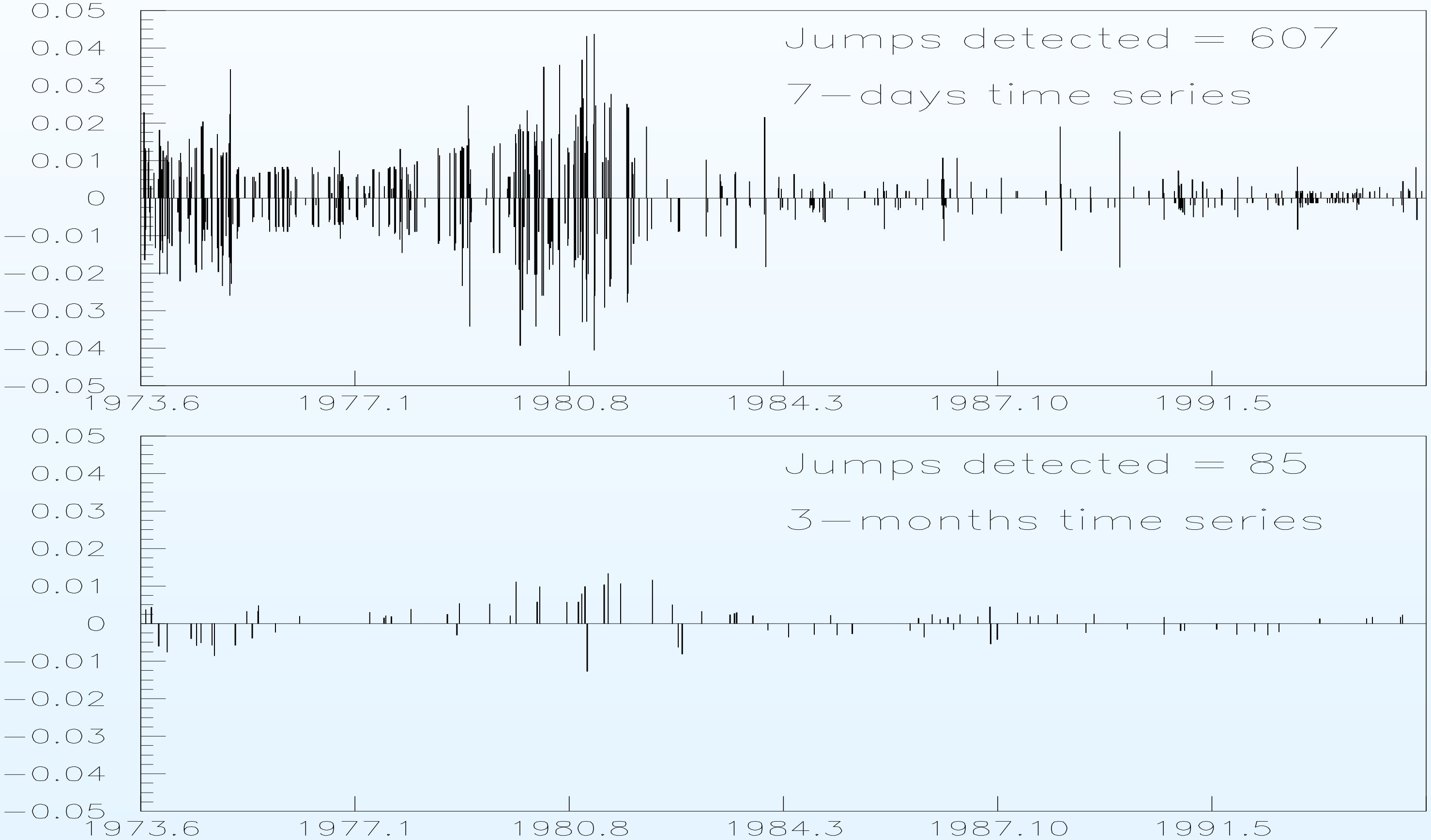
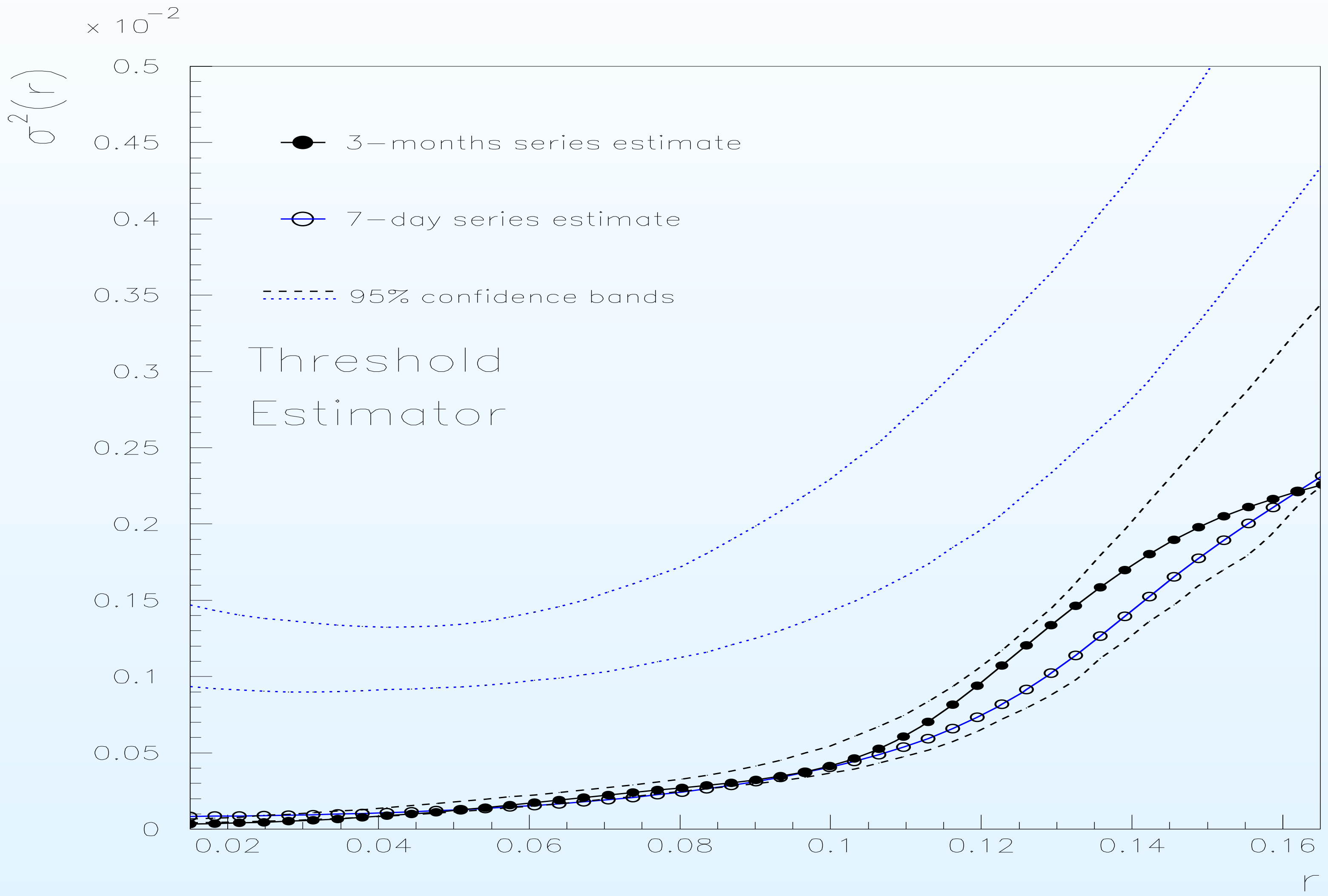


Estimation with the threshold estimator



Comparison with Bandi-Nguyen-Johannes

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- Using the fourth and the sixth moment, and assuming normality for C, with $\mathbb{E}[C] = 0$ and denoting by $\sigma_J^2 = \mathbb{E}[C^2]$, we have:

$$\lim_{\delta \to 0} \frac{1}{\delta} \mathbb{E}_t \left[(X_{t+\delta} - X_t)^4 \right] = 3\lambda(X_t)\sigma_J^4$$

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- After estimating $\lambda(X_t), \sigma_J$ we have:

$$\lim_{\delta \to 0} \frac{1}{\delta} \mathbb{E}_t\left[(X_{t+\delta} - X_t)^2\right] = \sigma(X_t) + \lambda(X_t)\sigma_J^2$$

Comparison with Bandi-Nguyen-Johannes

- Moment estimators can be obtained using:

$$\frac{1}{\delta}\mathbb{E}_t\left[\left(X_{t+\delta}-X_t\right)^k\right]=\frac{n\sum_{i=1}^{n-1}K\left(\frac{X_i-x}{h_n}\right)\left(X_{i+1}-X_i\right)^k}{T\sum_{i=1}^{n-1}K\left(\frac{X_i-x}{h_n}\right)}$$

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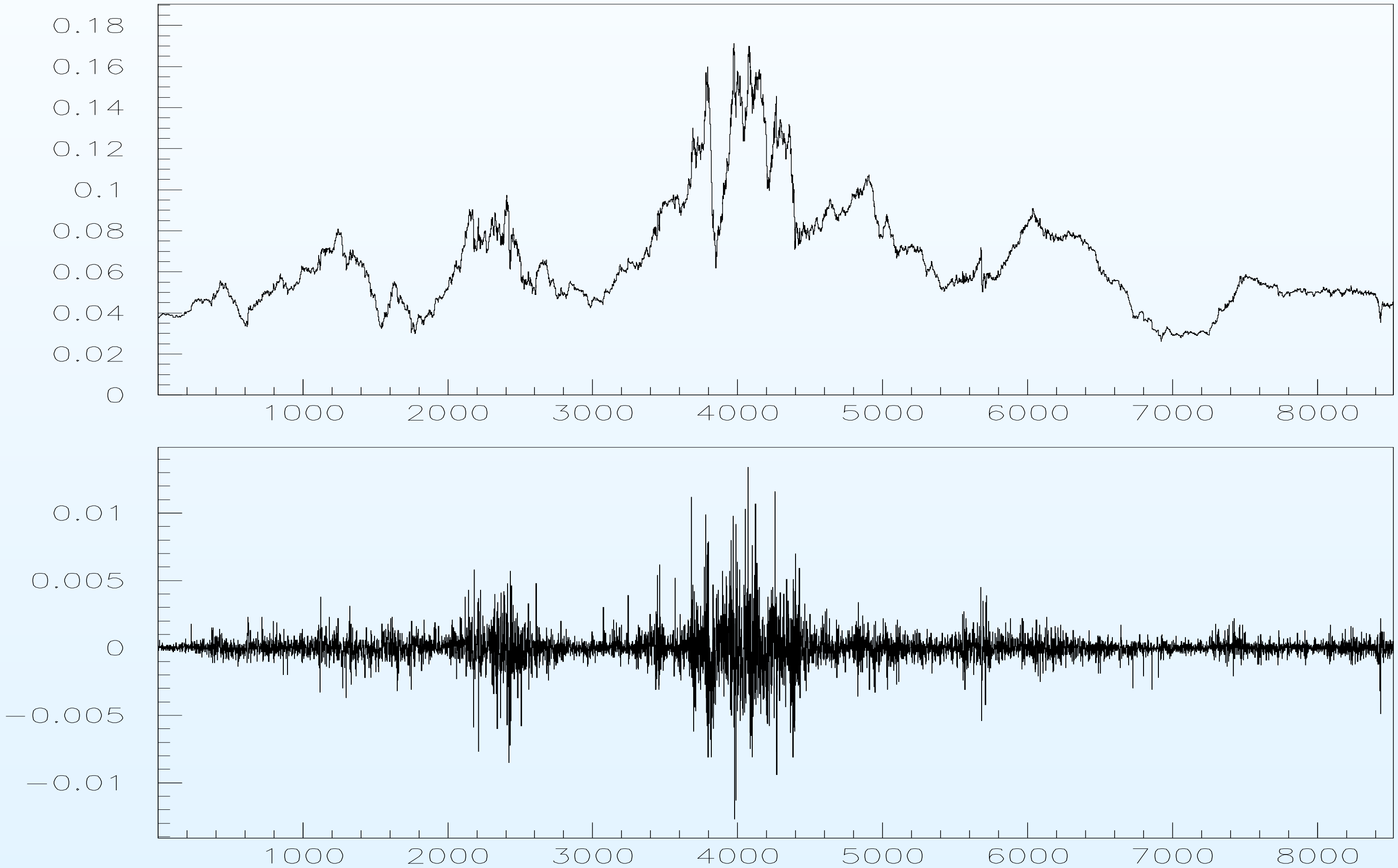
- Full limit theorems are developed in Bandi and Nguyen (2003), estimation on interest rate data is accomplished in Johannes (2004).

Johannes (2004) dataset



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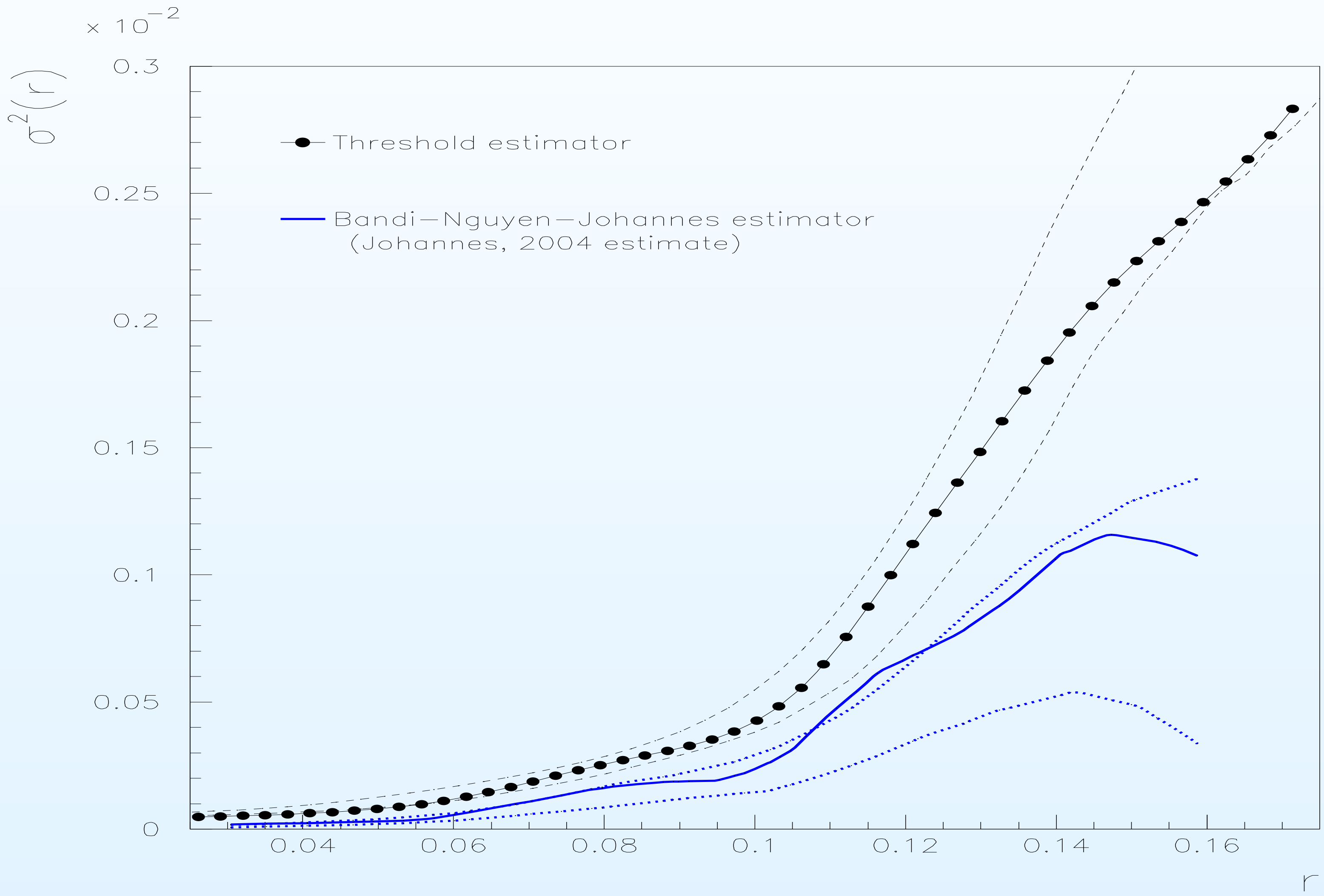

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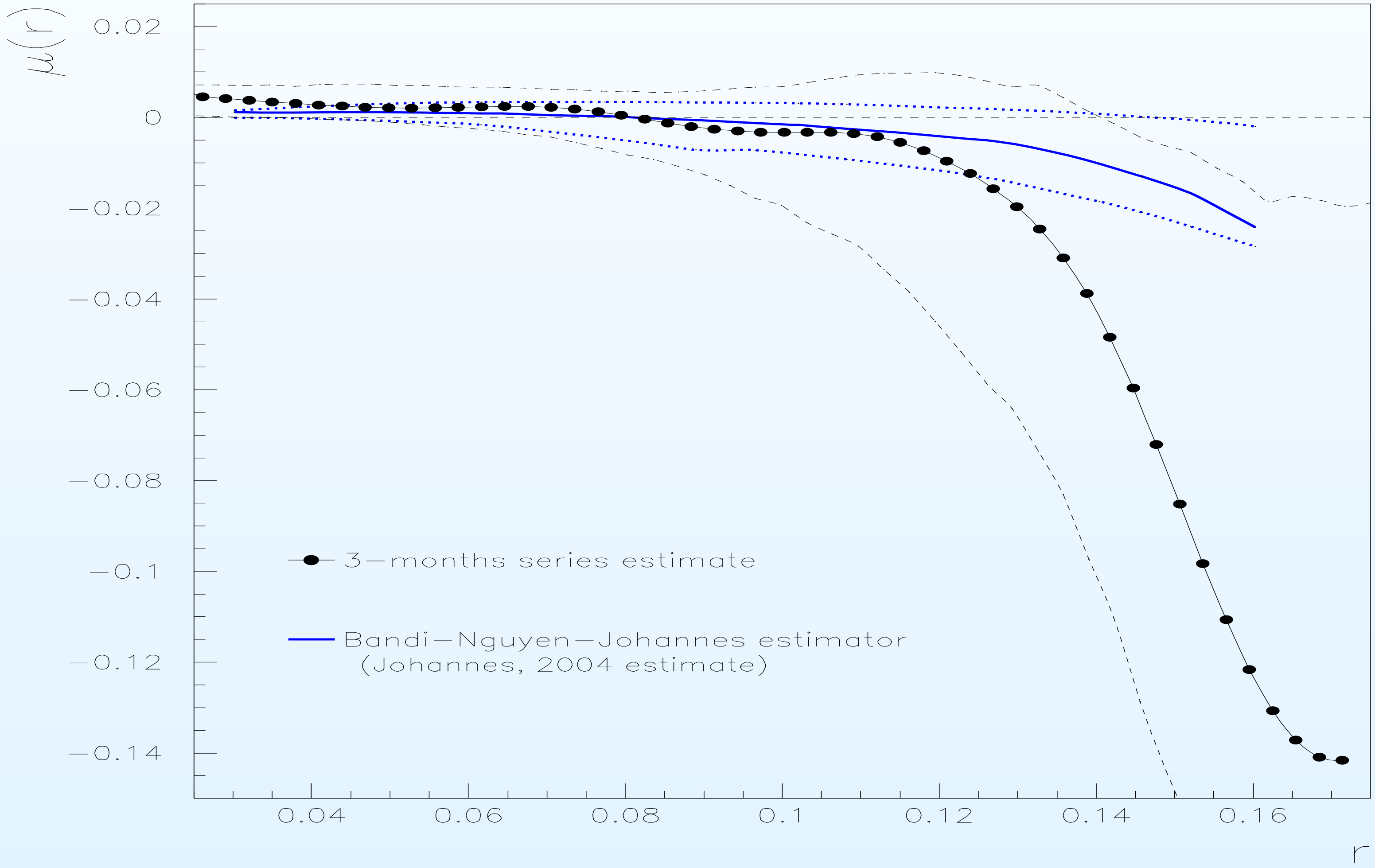

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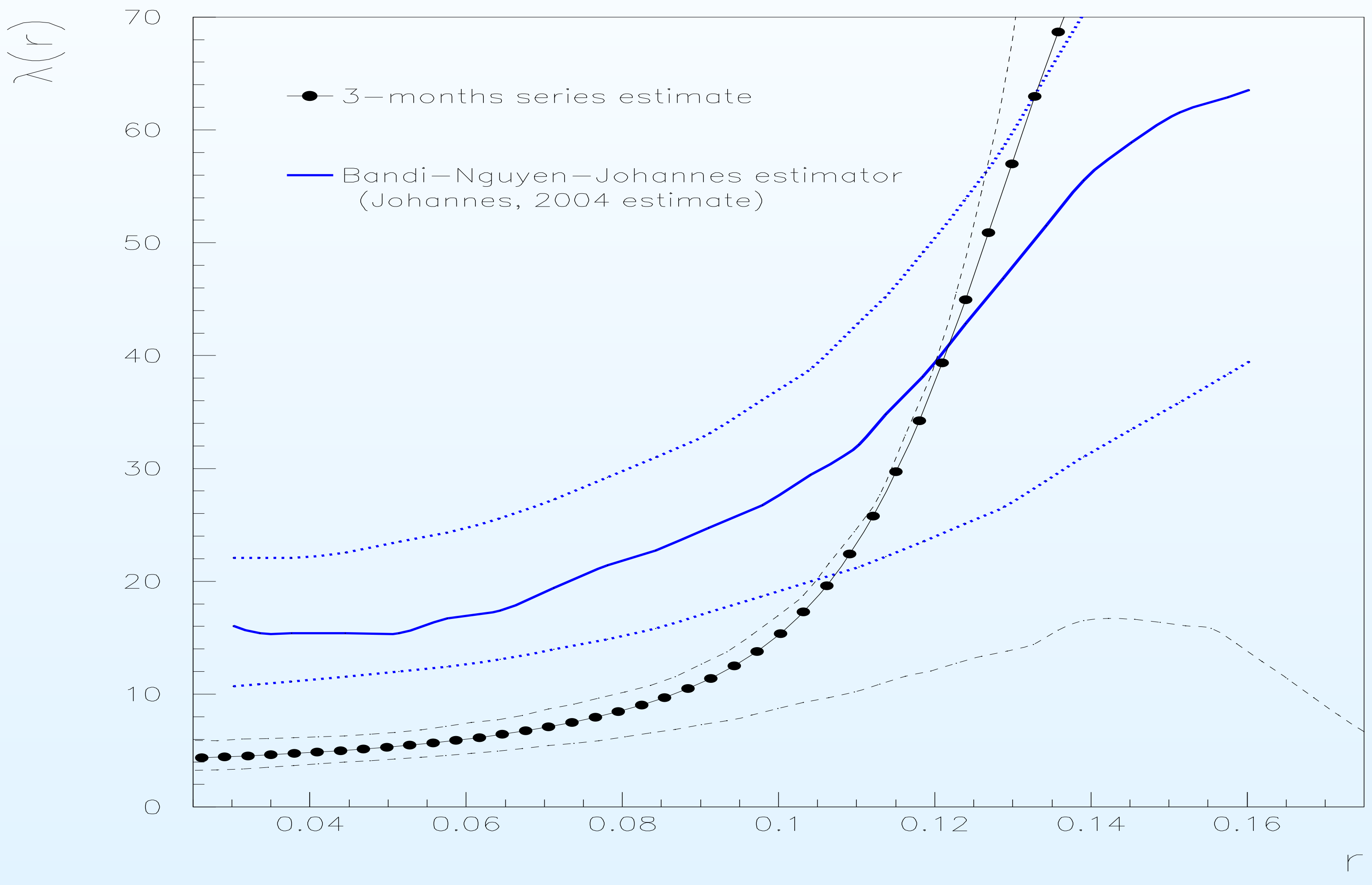
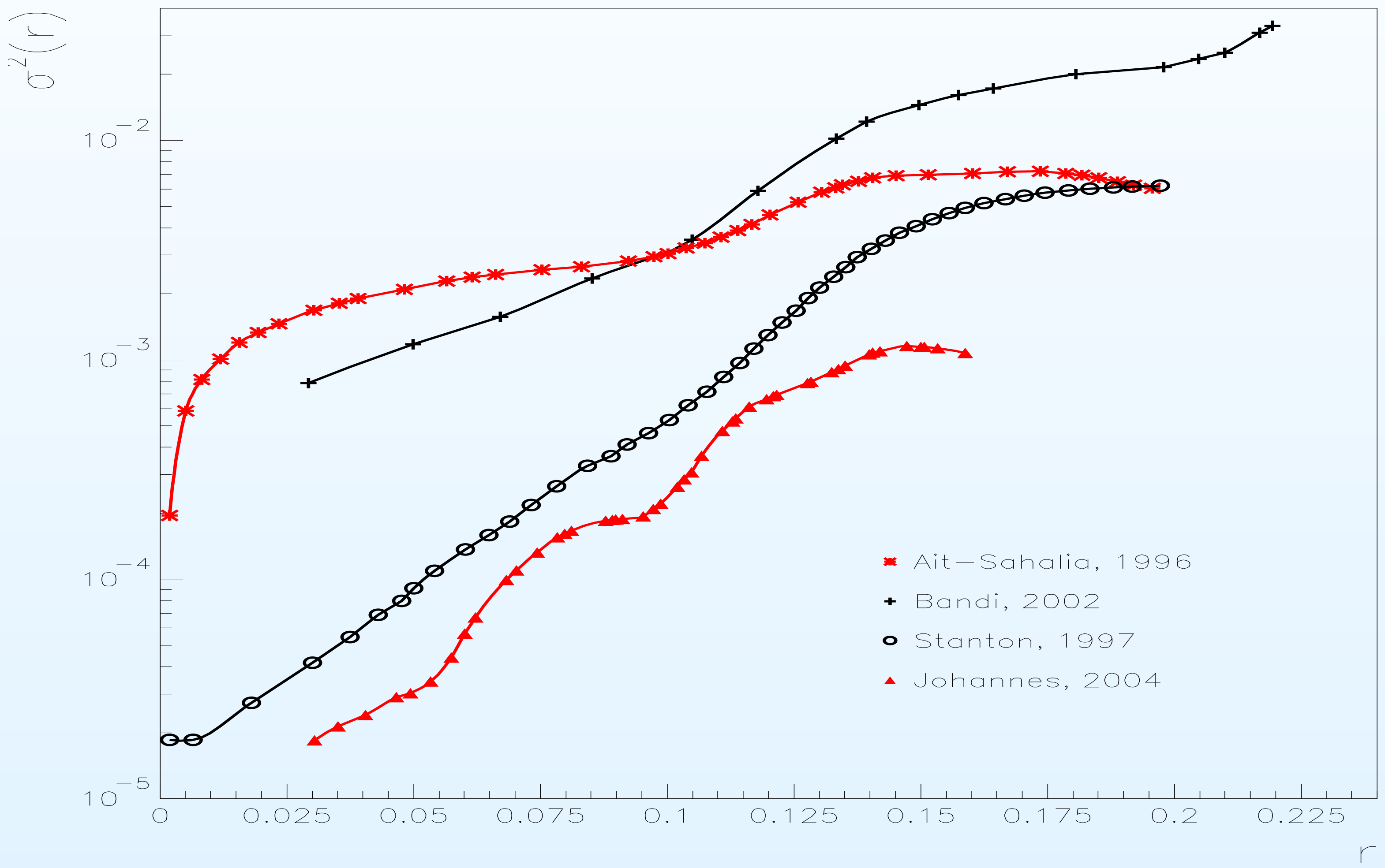
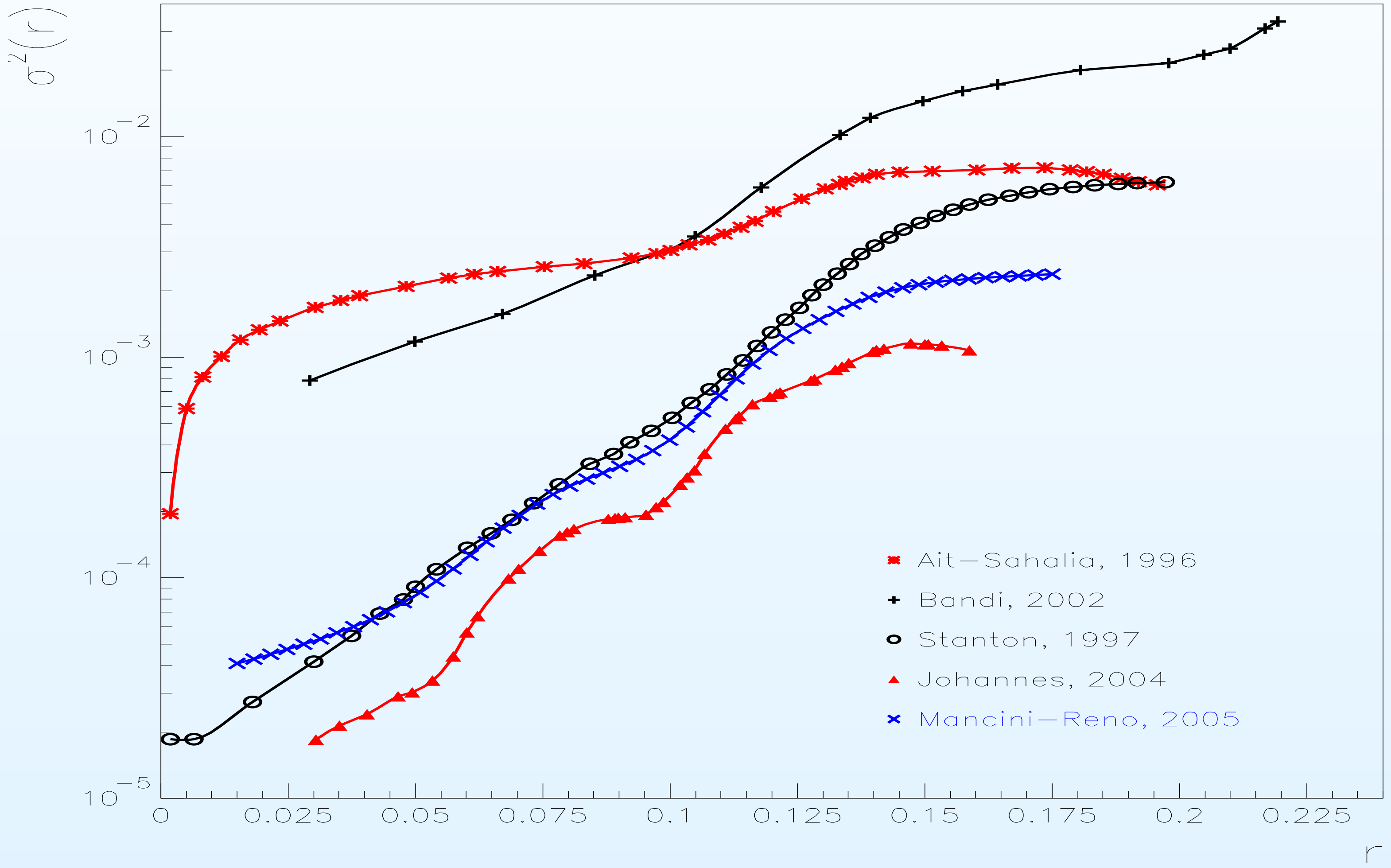


Comparison with the literature



Comparison with the literature



Conclusions

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- We show that the threshold can be set using standard econometric techniques (GARCH modelling)
- Monte Carlo simulations show that the proposed estimator of the diffusion coefficient is superior to alternatives which do not consider jumps
- We show that the 7-day time series is characterized by an inherent jump process which is absent on the 3-months time series. This is due to liquidity reasons.
- Our estimators provide different results than those obtained so far on jump-diffusion models: more diffusive variance and less jumps.

Work in progress

1. Explain the difference with Bandi-Nguyen-Johannes estimator

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   SV1FJ model - One day one jump (Pirino and Renò, 2007)

	Mean(%)	Standard Dev.(%)
BPV	27.72	37.77
TBPV	-7.91	18.75
C-TBPV	3.85	18.92
QV	53.71	113.41
TQV	-13.36	40.41
C-TQV	7.72	46.17
TriV	105.62	243.84
TTriV	-12.71	38.73
C-TTriV	10.36	46.94

Work in progress

1. Explain the difference with Bandi-Nguyen-Johannes estimator
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3. Apply above results to volatility forecasting, extending the HAR-RV model of Corsi (2004).

Extra-figures

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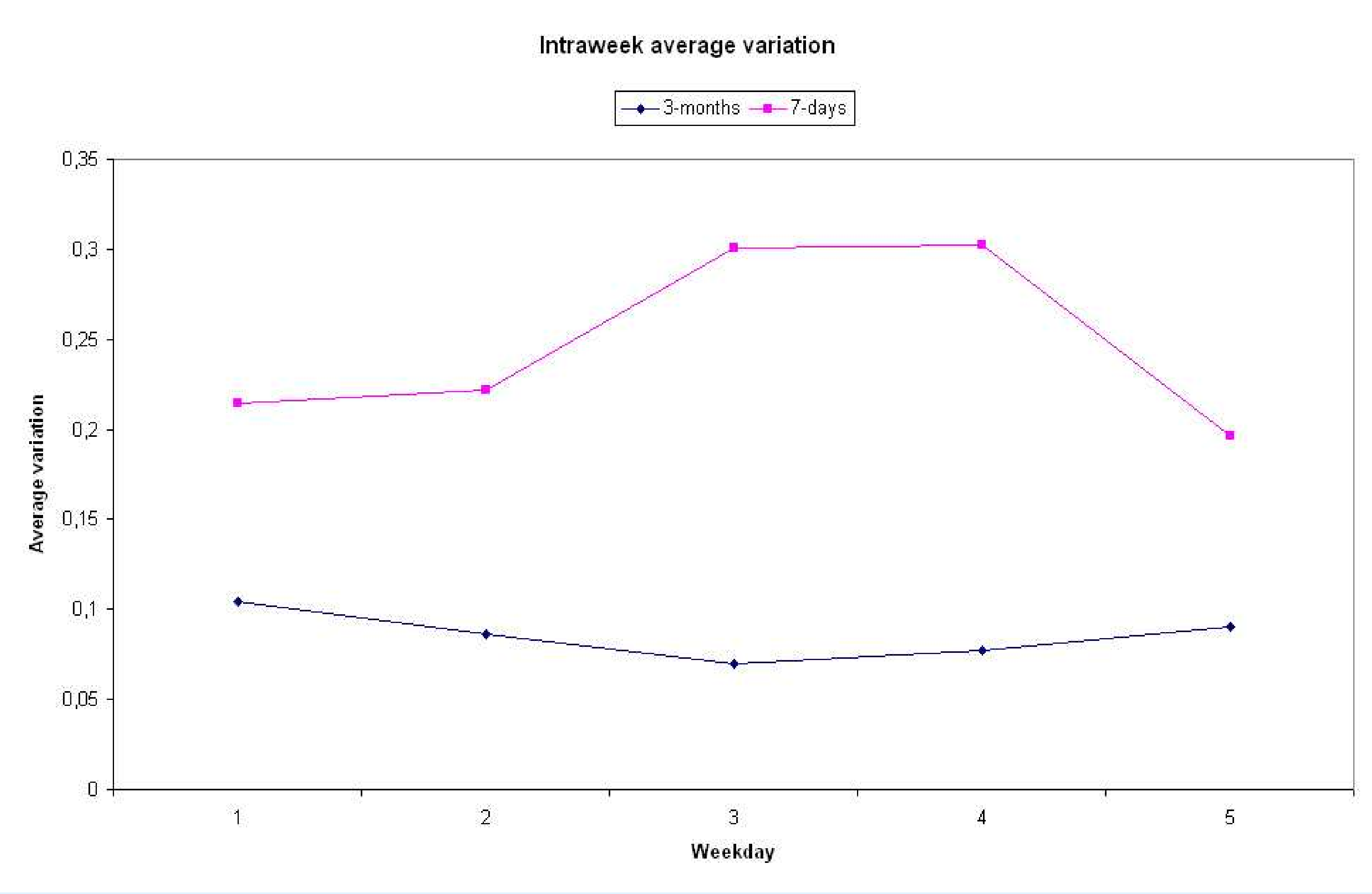

Extra-figures