Is volatility lognormal? Evidence from Italian futures

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Abstract

We study the unconditional volatility distribution of the Italian futures market, measuring it via Fourier analysis. Our data set consists of all tick-by-tick transactions in 2000 and 2001, a period characterized by unusually high volatility levels in its final part, because of the dramatic events following September, 11^{th} 2001. Our results show that the standard assumption of lognormal unconditional volatility has to be rejected for such a turbulent sample, since it is unable to capture the tail behavior of the distribution; a much better description is provided by a Pareto tail.

1 Introduction

Recent literature concentrated on the unconditional distribution of the fluctuations of financial asset prices. Volatility plays a crucial role in derivative pricing and risk management. Results in [9,10,13,1,2,7] among others show that the lognormal distribution fits very well the volatility distribution, much better than the Normal. Anyway, none of these studies focuses on the distribution of the tail, probably because of the scarcity of extreme events. On the other hand, results in [4,3] points out that standardized returns can fairly be approximated by a Normal distribution. This fact has particular theoretical importance, since many popular models, such as the ARCH family, rely on the assumption that standardized returns are normally distributed.

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In this note, we compute volatility for a data set with unusually high volatility, i.e. the futures price of the Italian stock index (FIB30) in 2000-2001. Many extreme events concentrate in late 2001 because of the dramatic events in United States. Our purpose is to verify whether the Log-Normal distribution for volatility and Normal distribution for standardized returns is still valid for the tails. Since we have at our disposal all tick-by-tick transactions, we adopt a volatility estimator which makes use of all observations with no need of aggregating or filtering data. This estimator is briefly described in Section 2. In Section 3 we show our results; Section 4 concludes.

2 Volatility measurement

In this Section we illustrate our volatility estimation technique: it makes use of an algorithm proposed in [11] and developed in [5,6]; this method is based on the Fourier analysis of the asset price time series, and it employs all the (unevenly spaced) observations.

If $S(t)$ is a generic asset price, let $p(t) = \log S(t)$ and require the quadratic variation of $p(t)$ to be bounded. We do the assumption that the behavior of the process $p(t)$ is described by the following stochastic differential equation:

$$dp = \mu(t,p)dt + \sigma(t,p)dW(t), \tag{1}$$

where $\mu(t,p)$ and $\sigma(t,p)$ are random functions and $W(t)$ is a Brownian motion. In this model $\sigma^2(t,p)$ is the instantaneous volatility at time t, i. e.

$$\sigma^2(t) = \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} E[(p(t+\varepsilon) - p(t))^2 | \mathcal{F}_t], \tag{2}$$

where $E[.|\mathcal{F}_t]$ denotes the conditional expectation operator with respect to the σ-field $\mathcal{F}_t$ generated by the full observation of the process until time t. We normalize the time window $[0,T]$ of the observations to $[0,2\pi]$. By means of the Fourier coefficients of dp

$$\begin{aligned}
a_0(dp) &= \frac{1}{2\pi} \int_0^{2\pi} dp(t) \\
a_k(dp) &= \frac{1}{\pi} \int_0^{2\pi} \cos(kt)dp(t) \\
b_k(dp) &= \frac{1}{\pi} \int_0^{2\pi} \sin(kt)dp(t) \qquad k \geq 1
\end{aligned} \tag{3}$$

we can obtain the Fourier coefficients of σ^2 through the following formulae [11]:

$$
\begin{aligned}
a_0(\sigma^2) &= \lim_{n\to\infty} \frac{\pi}{n} \sum_{s=1}^{n} \frac{1}{2}[a_s^2(dp) + b_s^2(dp)] \\
a_k(\sigma^2) &= \lim_{n\to\infty} \frac{2\pi}{n} \sum_{s=1}^{n} a_s(dp) a_{s+k}(dp) \\
b_k(\sigma^2) &= \lim_{n\to\infty} \frac{2\pi}{n} \sum_{s=1}^{n} a_s(dp) b_{s+k}(dp) \qquad (4)
\end{aligned}
$$

Then the Fourier-Fèjer inversion formula allows us to reconstruct $\sigma^2(t)$ in $[0, 2\pi]$:

$$
\sigma^2(t) = \lim_{n\to\infty} \sum_{k=0}^{n} \left(1 - \frac{k}{n}\right) [a_k(\sigma^2)\cos(kt) + b_k(\sigma^2)\sin(kt)]. \qquad (5)
$$

Given a time series of N observations $(t_i, p(t_i))$, $i = 1, \ldots, N$, we compress the data in the interval $[0, 2\pi]$; to connect the data we do the choice $p(t) = p(t_i)$ in $[t_i, t_{i+1}]$ (piecewise constant) [2]. The integral in equation (3) becomes

$$
\begin{aligned}
a_k(dp) &= \frac{1}{\pi}\int_0^{2\pi} \cos(kt)dp(t) = \frac{p(t_N) - p(t_1)}{\pi} - \frac{k}{\pi}\int_0^{2\pi} \sin(kt)p(t)dt = \\
&= \frac{p(t_N) - p(t_1)}{\pi} - \sum_{i=1}^{N-1} \frac{k}{n} \int_{t_i}^{t_{i+1}} \sin(kt)p(t)dt = \\
&= \frac{p(t_N) - p(t_1)}{\pi} - \sum_{i=1}^{N-1} \frac{p(t_i)}{\pi}[\cos(kt_i) - \cos(kt_{i+1})]. \qquad (6)
\end{aligned}
$$

We are interested to get the integrated volatility between 0 and 2π, given by

$$
\int_0^{2\pi} \sigma^2(t)dt = 2\pi a_0(\sigma^2). \qquad (7)
$$

The efficiency of the Fourier method has been assessed in [5,6]; by using simulations of the continuous time GARCH model, they show that the method

[2] Alternatively we could implement the Fourier method with linearly interpolated observations. In [5] it is shown that in this case a strong downward bias arises, increasing with sampling frequency.

performs well in computing integrated volatility, and they find that the forecasting performance of the GARCH model is improved with respect to what is established when classical methods are adopted.

3 Data analysis

Our data set consists of all transactions of the Italian FIB30, which is the future contract on the Italian stock index (MIB30), a weighted average of the 30 most capitalized stocks of the Italian stock market. Futures are delivered quarterly; we always use the next-to-expiration contract. If a simple cost-of-carry model for future price holds, and interest rates and dividends are constant within the time interval considered (one hour), the volatility of futures price is the same of stock price. Our data set ranges from January, 2000 to December, 2001 and the total number of records in our sample is about 5,480,000.

We measure hourly volatility via formula (7) after substituting the limit with a finite sum, setting:

$$\hat{\sigma}_t^2 = \int_t^{t+2\pi} \sigma^2(s)ds = \frac{2\pi^2}{N} \sum_{s=1}^{N} \frac{1}{2}[a_s^2(dp) + b_s^2(dp)], \qquad (8)$$

where the time unit 2π stands for one hour. To obtain this estimation, it is important the choice of N in equation (8). A priori we could choose a frequency N arbitrarily large (within the limits imposed by the actual frequency of the data), but microstructure effects such as the *bid-ask spread* may produce biases and measurements errors at very high sampling frequencies. Figure 1 shows the average over the whole sample of daily volatility measurement as a function of N; we choose to adopt $N = 300$, as a result of the tradeoff between increasing precision and avoiding distortion which clearly appear for larger values of N; in particular, the upward bias for very large N is due to spurious negative autocorrelation in tick-by-tick data [5]. We also repeated our analysis with $N = 100$ with no differences in the results. The value $N = 300$ for daily volatility corresponds to $N = 36$ for hourly volatility. This frequency corresponds to a time scale of nearly a minute. In total, we have 8 hourly observations per day (we use data from 9.15 to 17.15), 498 days in our sample, so we have 3984 volatility observations.

Figure 2 shows the average intraday pattern of daily volatility. It shows the typical $U-$shape pattern, together with spikes due to the release of public information and the opening of U.S. markets [4]. These patterns have been interpreted in the literature as deterministic; thus we detrend them by apply-

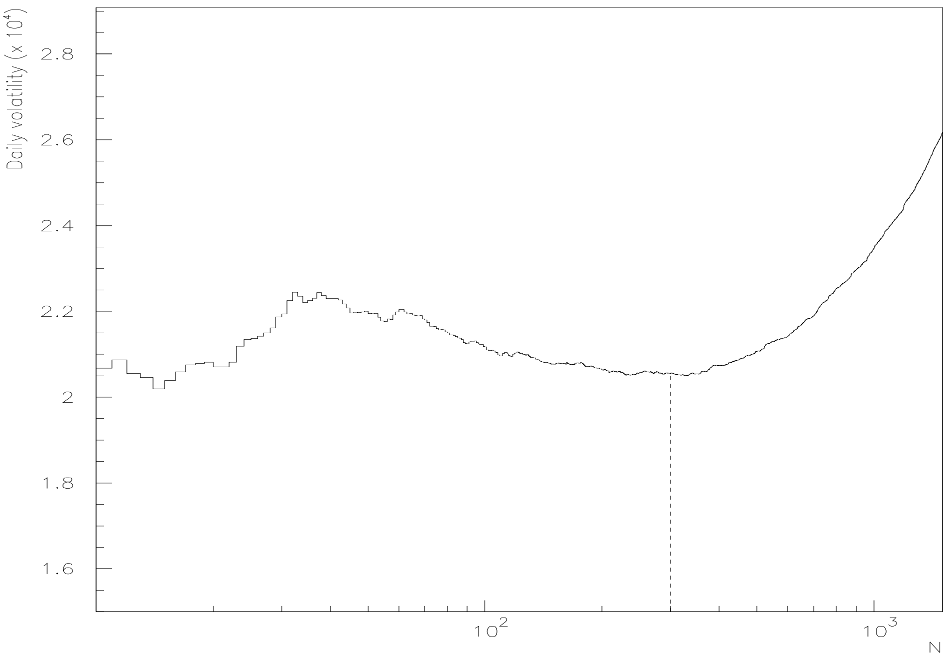

Fig. 1. Average daily volatility $\hat{\sigma}^2$ as a function of the cut frequency N in equation (8). We choose $N = 300$, as indicated by the dashed line, to measure volatility.

ing to our time series the following transformation: we reweight our volatility measurements through the following multiplicative terms [4,13]:

$$\lambda_j = \frac{1}{M} \frac{\sum_{k=0}^{M-1} \hat{\sigma}^2(j+8k)}{< \hat{\sigma}^2 >} \qquad j = 1, \ldots, 8. \tag{9}$$

where $\sigma^2(i)$ is the integrated volatility in the time interval i, M is the number of trading days considered, and $< \sigma^2 >$ is the average volatility over the full sample. Each coefficient λ_j is the average volatility in the hourly period j; dividing volatility by the multipliers we remove the daily periodicity. The reported results are obtained computing the weights λ_j excluding the days after September, 1^{st} 2001; anyway, we obtained almost identical results using all days to compute the weights.

Figure 3 shows the detrended time series of hourly volatility. Many sudden spikes are present; in particular, Figure 3 shows the market reaction to two sadly well-known events; September 11^{th} 2001, and November 12^{th} 2001, when an airplane crash on New York suburbs was, in the very first hours, interpreted as a new act of terrorism. The outliers caused by these events influence strongly the statistical properties of volatility distribution. To study the incidence on

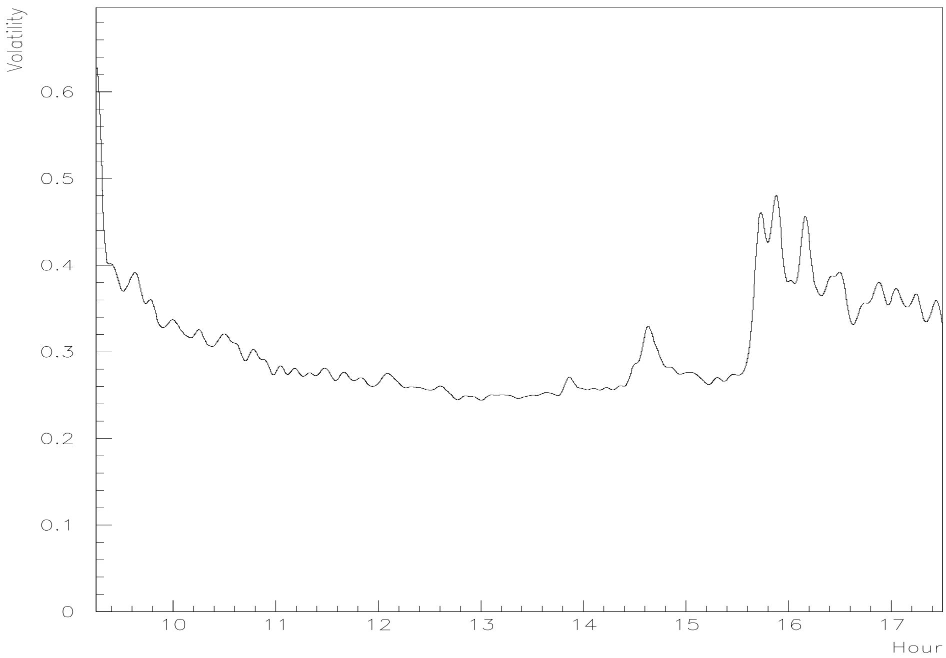

Fig. 2. Intraday volatility pattern, averaged over the full sample.

Table 1
Summary statistics for $\log \hat{\sigma}_t^2$. Limit values for the Jarque-Bera test are 5.99 (95%), 9.21 (99%).

$\log \hat{\sigma}_t^2$	raw	raw - 14 days	detrended	detrended - 14 days
Mean	-11.196	-11.252	-11.106	-11.163
Variance	0.9173	0.8073	0.7061	0.5963
Skewness	0.2916	0.0721	0.4986	0.1992
Kurtosis	3.29	2.82	3.84	3.17
Jarque-Bera	70.032	8.56	282.30	30.35

final results of such extreme observations, we do an alternative analysis excluding the data of the 14 trading days of the period 11/9/2001 - 28/9/2001, in which are concentrated most of the outliers.

Table 1 reports some summary statistic on the distribution of $\log \hat{\sigma}_t^2$, on raw and detrended data, and with and without the 14 days mentioned earlier. The skewness and kurtosis estimates are similar to those obtained by previous studies, which use a different method to compute volatility. The estimates of [4] for the FTSE-100 futures from 1990 to 1998 are 0.44 and 3.71; for the DJIA index from 1993 to 1998 the results in [8] are 0.75 and 3.78; the estimates in

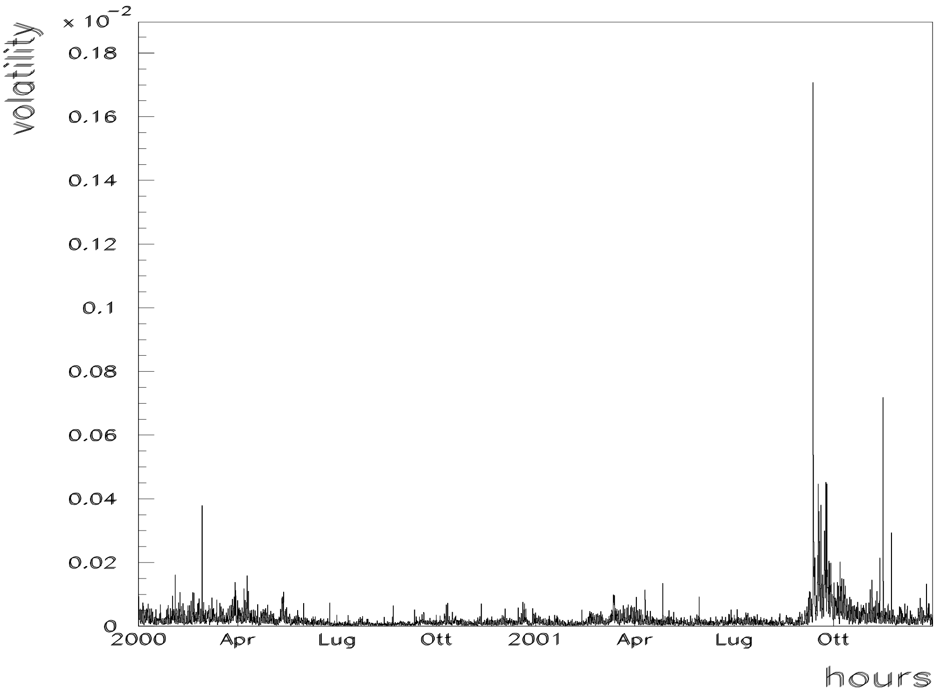

Fig. 3. Time series of hourly volatility from January, 11^{th} 2000 to December, 29^{th} 2001.

[2] are 0.35 and 3.27 for the DM/$ rate and 0.28 and 3.47 for the Yen/$ rate; similar results are obtained in [1] on individual stocks. It is worth to note as the variance of volatility decreases when detrending from 0.9173 to 0.7061; not surprisingly, when removing the 14 days, this variance decreases even more, to 0.5963.

The skewness and kurtosis of detrended data is quite far from the Normal values of 0 and 3. We test for normality making use of the classical Jarque-Bera test, which is a joint test on skewness and kurtosis which is distributed as a chi-square with 2 degrees of freedom. This test clearly rejects the hypothesis of normality on the whole sample. Tail events are clearly responsible for that; if we remove the 14 days after September, 11^{th}, the skewness steps down from 0.4986 to 0.199, the kurtosis decreases from 3.84 to 3.17. Anyway, the Jarque-Bera test still rejects the normality hypothesis for this reduced sample. Interestingly enough, if we do not detrend data and exclude the 14 days, we cannot reject lognormality at the 99% confidence level. This is in agreement with the results in previous literature, which document lognormally distributed volatility in quiet periods: see also the discussion in [12].

We now concentrate on the tail of the distribution. Figure 4 shows the empirical cumulative distribution function $F(x) = \int_x^\infty f(s)ds$, where $f(s)$ is the density. We obtain it by sorting the observations of volatility σ_j in descending

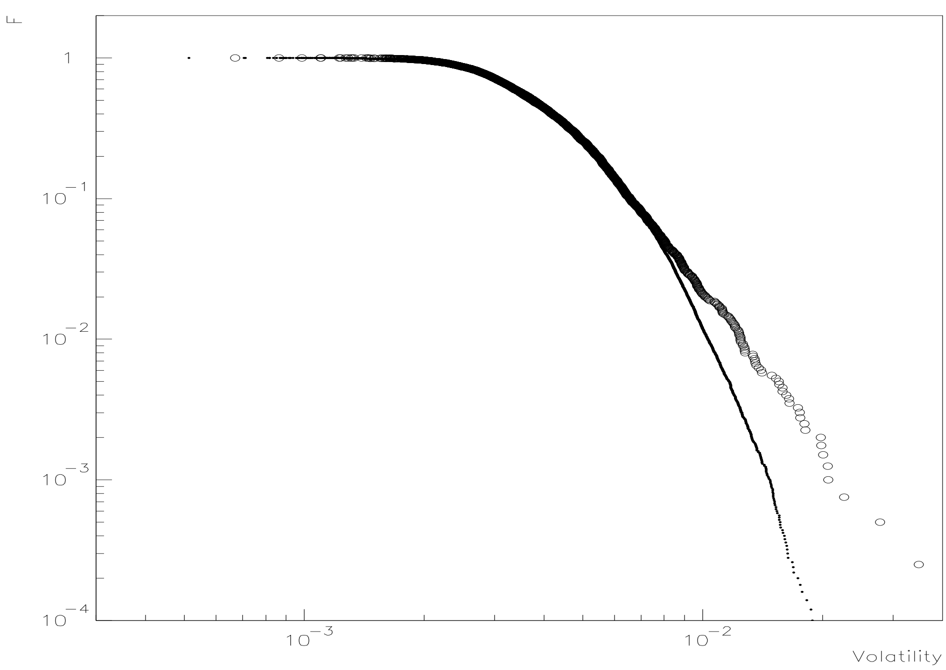

Fig. 4. Cumulative distribution of $\hat{\sigma}_t$ (white dots), together with the cumulative distribution of a lognormal density with the same mean and variance (black dots).

order and express the cumulative distribution in terms of the sorted data as

$$F(\sigma > \sigma_j) = \frac{j}{N} \qquad j = 1, \ldots, N \tag{10}$$

where N is the total number of observations. Figure 4 also shows the cumulative distribution function of the lognormal distribution with the same mean and variance of the data, obtained via Monte Carlo simulation; it is clear that the empirical distribution is more "fatted" than the theoretical one. We observe that the tail of $F(x)$ is very well modeled via a Pareto tail, i.e. $F(x) \sim x^{-\alpha}$. This is in contrast with the assumption of lognormality, whose cumulative distribution function drops to zero faster, in particular faster than $x^{-\alpha}$ for every α. Estimates of α are very important to assess the finiteness of the second moment of the distribution: $0 < \alpha < 2$ would imply a stable law. Our estimates, obtained with the Hill estimator, yield $\alpha = 3.62 \pm 0.16$; this value has been obtained with the largest 500 observations, but it is fairly robust to this choice. This is consistent or somewhat larger than estimates obtained on the S&P 500 index by [10]: they find the value $\alpha = 3.10 \pm 0.08$. In any cases, our estimate rules out the possibility of stable law.

Table 2 reports the same summary statistics on the distribution of standardized returns, again on raw and detrended data, and with and without the 14

Table 2
Summary statistics for $\frac{r_t}{\hat{\sigma}_t}$. Limit values for the Jarque-Bera test are 5.99 (95%), 9.21 (99%).

$\frac{r_t}{\hat{\sigma}_t}$	raw	raw - 14 days	detrended	detrended - 14 days
Mean ($\times 10^3$)	5.98	6.15	2.43	1.88
Variance	0.9138	0.9113	0.9416	0.9377
Skewness	0.0043	0.0091	-0.036	-0.034
Kurtosis	2.87	2.86	3.44	3.41
Jarque-Bera	2.67	3.11	33.76	27.72

days mentioned earlier. Again we get results which are quite in line with the literature. In particular, the skewness and the kurtosis are very close to the Normal values, and the Jarque-Bera test does not reject the hypothesis of normality for raw data, even for the reduced data sample. What is worth to note, is that when detrending volatility, the Jarque-Bera test rejects the hypothesis of normality. Anyway, it is hard to interpret this result since the rejection is not so sharp, and especially it is not robust to detrending.

4 Conclusions

This note is devoted to the study of the unconditional distribution of integrated volatility. In particular, we concentrate on the hypothesis of a lognormal distribution for modeling volatility, and normal distribution for standardized returns. To this purpose, we use two years of all recorded transactions on the Italian stock index future, and we adopt a volatility estimator which is well suited to the time structure of tick-by-tick observations. The years selected, 2000 and 2001, are full of incertitude on international markets, and consequently contain a lot of large volatility observations.

We show that the lognormal distribution has to be rejected, on the basis of classical normality tests, as a model for the unconditional distribution of hourly volatility; we show that the main motivation for that failure is in the tail of the distribution. When removing the largest observations, the lognormal distribution turns out to be a much more reasonable description. We then conclude that the assumption of lognormal volatility is not valid for particularly turbulent periods.

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