On the presence of unspanned volatility in European interest rate options*

Roberto Renò†

Università di Siena,

Dipartimento di Economia Politica,

Piazza S.Francesco 7, 53100, Siena, ITALY

email: reno@unisi.it

Adamo Uboldi

Istituto per le Applicazioni del Calcolo, CNR,

Viale del Policlinico 137, 00161, Roma, ITALY

email: uboldi@iac.rm.cnr.it

August 30, 2004

Abstract

In a recent paper, Collin-Dufresne and Goldstein (2002) show that the movements of the yield curve and of interest rate derivatives are mostly uncorrelated, advocating the presence of unspanned volatility. In this letter, we show that their results can be explained in the framework of a Gaussian HJM model with humped term-structure volatility. This implies that hedging interest rate derivatives with interest rate swaps is not ruled out.

1 Introduction and Motivation

Interest rate diffusions are nowadays fundamental tools for derivative pricing. No-arbitrage modeling looks for consistency between interest rate and derivative price movements. Despite a growing literature, a satisfactory model which accounts for both bond and interest rate derivative pricing has not been found.

In this letter, we concentrate on the recent results of Collin-Dufresne and Goldstein (2002), who advocate the use of affine models (Duffie and Kan,

*We gratefully acknowledge Flavio Angelini and Stefano Herzel for providing pricing routines as well as for stimulating discussions. We also thank the participants of the V Workshop of Finance held in Siena, in January 2004. All errors and omissions are our own responsibility

†Corresponding author

1996) displaying unspanned volatility. The main empirical motivation comes from the regression analysis they perform on the movements of option prices (specifically straddles, portfolios of caps and floors) against the movements of the yield curve (specifically swap rates). Caps, floors and swaps are the most intensively traded interest rate sensitive instruments. On a data set on US, UK and Japan markets in the period 1995-2000, they regress straddle movements against swap movements. It is worth noting that straddles are very sensitive to interest rate volatility, more than on downward or upward movements. If the movements of the yield curve explain the movements of option prices, this regression should yield an R^2 close to unity. They find a puzzling result. On average, only 25% of straddle variations is explained by swap variations. The Authors comment in the following way. First, term structure movements are not correlated with the movements of cap and floor prices. Second, even if the regressions are affected by strong collinearity, R^2 still retains its interpretation of being the maximum percentage of caps-floors replication that can be accomplished using swaps. Thus, hedging is only partially possible. They suggest a theoretical framework in which the state variables are driven by affine diffusions. One of this state variables is the volatility of the short rate. To explain their results, the short rate volatility should have a very low impact on bond prices, and a strong impact on derivative prices. They call this volatility *unspanned*, that is derivatives

Table 1: R^2 of the regression of straddle variations against swap variations (CG) and of straddle variations against HJM straddle price variations (HJM).

Maturity	1y	2y	3y	4y	5y	7y	10y
R^2 (CG)	0.3918	0.6195	0.7280	0.7895	0.8166	0.8436	0.8561
R^2 (HJM)	0.8209	0.9123	0.9166	0.9227	0.9151	0.9026	0.9072

volatility that cannot be hedged via yield curve instruments.

We started by repeating the same exercise on a data set on Euro swap and Euribor rates, ranging from January 1999 (start of EMU) to December 2001, for a total of 781 days. We have Euribor rates at 3,6 and 9 months, and swap yearly from 1 to 10 years, then 12, 15, 20, 25 and 30 years. Moreover we have daily at-the-money caps and floors in the same period, with maturities of $1, 2, 3, 4, 5, 7, 10$ years.

Our results are displayed in Table 1, labeled CG, and they are in line with those of Collin-Dufresne and Goldstein (2002), even if our R^2 are larger, probably because of the use of daily data instead of monthly data, which implies smaller variations and larger R^2 for our data set. However, we find poor performance at short maturities, and the performance is increasing with maturity.

Our feeling about these results is that in the runned regressions we are severely misspecifying the error term. It is well known that the interest rate volatility displays a hump-shaped behavior with maturity, ignoring which could lower the R^2 of the regression. We then estimate an HJM model for the term-structure with the purpose of describing the volatility of the yield curve, and see the impact on the results in Table 1.

2 The model

In this work, we focus on the well-known Heath-Jarrow-Morton framework, proposed in Heath et al. (1990, 1992). The HJM framework is a general arbitrage free framework for the term structure of forward rates. It is based on the following equation for the forward rates:

$$df(t,T) = \mu(t,T,f(t,T)) + \sum_{i=1}^{N} \sigma_i(t,T,f(t,T))dW_i(t), \tag{1}$$

where t denotes time, T maturity, $f(t,T)$ is the forward rate and $W_i,\ i = 1,\ldots,N$ are independent Brownian motions under the risk neutral probability; σ_i are N general functions. In order to avoid arbitrage, the following relation between drift and volatility has to be imposed:

$$\mu(t,T,f(t,T)) = \sum_{i=1}^{N} \sigma_i(t,T,f(t,T)) \int_t^T \sigma_i(t,u,f(t,u))du. \tag{2}$$

It follows that the drift in equation (1) is completely specified when the number of factors and the corresponding volatility functions are.

In what follows, by an HJM model we mean a precise specification of the number and shape of volatility functions.

We restrict our choice to a *Gaussian* HJM model, i.e. we drop the dependence on f in σ_i. With this choice of σ_i we can provide, as in Brace and Musiela (1997), closed analytical formulas for the drift and for derivative prices, such as caps, floors and swaptions, see also Brace and Musiela (1994).

The main drawback of the Gaussian choice is that it allows negative rates, a feature which, however, is not ruled out by data and which is not as crucial as sometimes stated in the financial literature. Most important, there is overwhelming evidence of stochastic volatility in interest rates, e.g. see Chapman and Pearson (2001), which cannot be accounted for with deterministic volatility, especially for hedging purposes, see the discussion in Dumas et al. (1998). We will comment on this later.

In addition to the previous simplifications, we adopt a constant maturity choice, letting the volatilities depend only on $T - t$.

Since the HJM model is factorial, it seems quite natural to extract the main factors driving the evolution of the term structure by principal component analysis (PCA); this approach goes back to Litterman and Scheinkman (1991), and has been largely used in applications similar to ours, see e.g. Driessen et al. (2000). It is well known indeed that few factors (a number of three can always be regarded as an excellent approximation) can explain a large component of the volatility structure across maturities, since the forward rates at different maturities are highly correlated.

We found, almost generally, that the factor loadings of the volatility matrix eigenvectors can be fitted with the same parametric representation given by:

$$\sigma_i(x) = (\alpha_i + \beta_i x)e^{\gamma_i x} + \delta_i, \tag{3}$$

where $x = T - t$.

Moreover the functional form (3) allows a simple analytical determination of derivative prices. The coefficients $\alpha_i, \beta_i, \gamma_i, \delta_i$ can be calibrated on the historical eigenvectors' factor loadings, using common procedures of minimization, such as gradient descent. A model quite similar to this can be found in Driessen et al. (2000); Fan et al. (2003). One-factor humped-volatility models are studied empirically in Moraleda and Vorst (1997); Ritchken and Chuang (1999). See also Angelini and Herzel (2002) for Gaussian models with consistent initial curves. Our model is the same as in Mercurio and Moraleda (2000) with one extra parameter.

3 Results and conclusions

We then fit the model to our data set. For the bootstrapping procedure needed to infer the forward rates, we adopt linear interpolation between

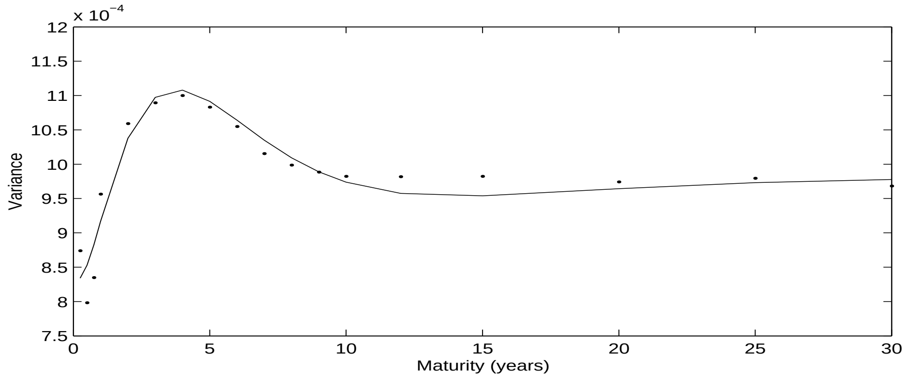

Figure 1: Observed forward rate variance (dots) vs. model fit (solid line)

swap rates of adjacent maturity.

We fit a three-factor model. The fit of the variance is shown in Figure 1, and it shows that the model was able to capture the humped shape.

We then regress the observed straddle movements against the straddle price movements implied by the fitted model. It is important to stress that model prices are a function of the term-structure only, thus we just performed a, somewhat complicated, change of variables. Results are displayed in Table 1, labeled HJM. We can see that the R^2 are now uniformly close to one across maturities, and that the model performed very well in modeling straddle prices. Interestingly, we get similar results if we use two factors instead of three, indicating that a three-factor model overfits the data. Figure 2 shows the time series of observed cap prices and model prices. Even if we do not get a completely satisfactory pricing, because of the inherent mis-

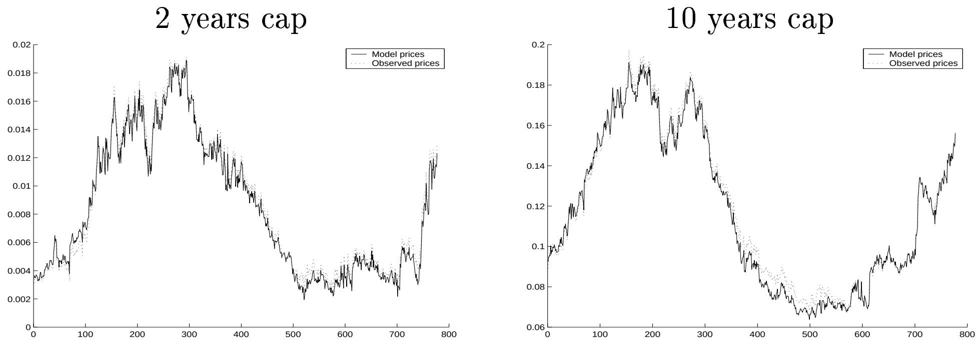

Figure 2: Cap prices (dashed) vs. model prices (solid) daily time series.

specification of Gaussian models, we observe an almost perfect correlation between the movements. The error in pricing levels is due to the absence of heteroskedasticity in our model, which is instead displayed by interest rates.

Our conclusions do not rule out the possible presence of unspanned volatility. It remains an intriguing and empirically testable theory, which provides a different explanation of the empirical stylized facts. HJM models still retain their pricing performance if volatility is carefully estimated. More important, we show that hedging derivatives with swap rates is possible. This result has also been confirmed for the swaptions market by Fan et al. (2003).

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